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LXIV. *Distribution of Energies of Electrons in Gases.* *By* J. S. TOWNSEND, *M.A., F.R.S., Wykeham Professor of Physics, Oxford* *.

1. THE conclusions with regard to the distribution of the energies of electrons which have been deduced from a study of the process of ionization by collision and of the radiation from uniform glows in discharge-tubes are in good agreement with the results of the investigations of the motion of electrons by the methods adopted in the kinetic theory of gases †. In these investigations the ordinary laws of probability are adopted, which show that there is a wide distribution of the energies of the electrons about the mean value; it may not be as wide as the Maxwellian distribution, but it is of that order.

The results of these investigations are generally accepted by students who are familiar with the ordinary laws of probability, but to some authorities who are said to be knowing and well skilled in this particular branch of natural knowledge the investigations appear to be obscure and open to suspicion.

There is not any scientific reason to support this view,

* Communicated by the Author.

† Phil. Mag. 6, xl. p. 505 (1920); and ix. p. 1145 (June 1930); 'Motion of Electrons in Gases' (Clarendon Press, Oxford, 1925).

and it is significant that those who adopt it also adopt without hesitation the laws of collisions discovered by Franck and Hertz.

2. According to these laws when a large number of electrons all with the same kinetic energy E_A move from a plane A in a gas to a parallel plane B under the action of an electric force perpendicular to the planes they all arrive at the plane B with the same energy E_B. Also, in agreement with Lenard, they found by their method of investigation that the free paths of electrons with velocities between 10 and 2 volts are very nearly the same as the free paths deduced from the kinetic theory of gases. Several experiments have been interpreted by these laws, where quite different conclusions would be obtained if it were admitted that the electrons lose different amounts of energy in the collisions with molecules.

There are many objections to these departures from well-established principles, as is shown by Bailey * in his paper entitled "Die Theorien von G. Hertz über Bewegungen langsamer Electronen in Gasen." He points out that the methods adopted by Franck and Jordan, relying on Hertz's theories, in order to show that the collisions between electrons and atoms are the same as between perfectly elastic spheres are scarcely even of qualitative significance.

There has been no answer to these objections, but recently Didlaukis † has published a theoretical investigation which has led him to a conclusion in agreement with Franck and Hertz. He claims to have shown that in the steady motion attained by electrons in a gas under the action of an electric force the distribution of the energies of the electrons is very narrow. He states that if E_1 be the mean energy of agitation there are practically no electrons with energies greater than $1{\cdot}06\ E_1$, or less than $\cdot 94\ E_1$, assuming that the mean free path is independent of the energy of the electron.

This result is shown by a curve which gives the distribution of the energies in hydrogen. He also states that the distribution is much narrower in gases of higher atomic weight, and concludes that in helium, neon, and some other gases the distribution of the energies must be extremely narrow.

* V. A. Bailey, *Zeit. für Physik*, 68, xi. & xii. p. 833 (Feb. 1931).
† M. Didlaukis, *Zeit. für Physik*, 82, xi. & xii. p. 709 (1933).

Before considering the possibility of a motion of this type it is of interest to indicate briefly the evidence, which shows that in all cases of steady motion in gases the energies of the electrons are widely distributed about the mean energy.

3. It has been fond by direct experiment that electrons attain a steady state of motion in a gas under the action of a uniform electric force, where the mean kinetic energy of the electrons is greater than the mean energy of agitation of the molecules of the gas *. The distribution of the energies of the electrons about the mean energy obtained with small forces was investigated by Pidduck †. He gave an accurate formula for the coefficient of diffusion, and also one for the velocity of the electrons in the direction of the electric force, in terms of the mean free path and the mean velocity of agitation of the electrons.

There is no simple expression for the distribution of the energies of the electrons, but it appears that a sufficient degree of accuracy is obtained in calculations where the mean energy of agitation of the electrons is not large compared with the energy of agitation of the molecules of the gas by taking the distribution to be the same as in the Maxwellian distribution. Also from the values of the velocities of the electrons which had been found experimentally he showed that the transfer of energy in the collisions of electrons with molecules of the gas (air) was nearly the same as in the collisions between elastic spheres.

4. There are many experiments which can only be explained on the hypothesis that the energies of the electrons are also widely distributed when the mean energy of the electrons is much greater than the energy of agitation of the gas. This is shown by the fact that ionization by collision is obtained under conditions where the mean energy of agitation of the electrons is small compared with the energy required to ionize the molecules ‡.

* Proc. Roy. Soc. A, lxxxi. p. 464 (1908).

† F. B. Pidduck, Proc. Roy. Soc. A, lxxxviii. p. 269 (1913); and Proc. Lond. Math. Soc. series 2, xv. pt. ii. p. 89 (1915).

‡ Phil. Mag. (6) xl. p. 505 (1920).

The difference between the mean energy of agitation and the energies of some of the electrons is very noticeable in the ordinary positive columns of discharge-tubes. Thus, in a wide tube containing helium at pressures of the order of 10 millimetres the mean energy of agitation may be less than 3 volts, while the radiation and the ionization required to maintain the current are due to collisions of electrons with atoms of the gas in which the energies of the electrons are greater than 20 volts.

The theory of the motion depends on the ordinary laws of probability, and it has been shown that as the electric force and the mean energy of agitation of the electrons increase, the distribution changes from the Maxwellian form to a form depending on the diffusion of the electrons in the electric field *.

5. The inequalities in the energies of electrons are caused by the differences in the changes of the energy of the electrons in collisions with molecules of the gas, and by the differences in the motion in the direction of the electric force.

For simplicity it is desirable to consider separately the distributions of the energies due to these two causes, and it is of interest to show that these distributions may be deduced from the results of experiments on diffusion by considering the distribution in space of the molecules of a small quantity of a gas G which diffuses into another gas. It may be supposed that the molecules of the gas G are initially at points on a plane A, and after a time they are distributed on each side of the plane. The distances x of the molecules of the gas G from the plane increase with the time, but the rate of increase diminishes as the time increases. If $\bar{x}^2$ be the mean square of the distances of the molecules from the plane A, the value of $\bar{x}^2$ at a time t is $2Kt$, where K is the coefficient of diffusion.

If N be the total number of molecules of the gas G initially at the plane A, dn the number between two parallel planes at the distances x and $x+dx$ from A, at the time t, the distribution is given by the equation

$$n/N=(4\pi Kt)^{-1/2}\epsilon^{-x^2/4Kt}. \quad . \quad . \quad . \quad . \quad (1)$$

Also it has been found that equation (1) applies to the diffusion of electrons when the electric force is in a direction perpendicular to the axis x.

* Phil. Mag. ix. p. 1145 (1930).

6. In this case the motion in the direction x is very simple, since the mass m of the electron is small compared with the mass M of a molecule of the gas, and all directions of motion of an electron after a collision are equally probable. If l be the mean free path of an electron, the coefficient of diffusion of electrons all with the same velocity U is lU/3.

Since the free paths are of different lengths the electrons collide with different numbers of molecules of the gas in the time t; but if these numbers are large the number of collisions c of each electron with molecules is approximately Ut/l.

Thus the quantity 4Kt becomes $4l^2c/3$, and the following equation is obtained for n in terms of x and c:

$$n/\mathrm{N}=\sqrt{3/4\pi l^2 c}\ \epsilon^{-3x^2/4l^2c}. \quad . \quad . \quad . \quad . \quad (2)$$

It may be assumed that the velocity U of the electron is large compared with the mean velocity of agitation of the molecules of the gas, and in order to simplify the investigation it may also be assumed that the free paths of the electrons are independent of velocity U.

Under these conditions if an electron starts from a plane A the probability of it being at a distance greater than x from the plane after traversing a number of free paths c is independent of the velocity U. This probability is represented by the integral $\int_x^\infty n\,dx/\mathrm{N}$, where n/N is given by equation (2).

7. In estimating the changes x in the positions of the electrons due to diffusion the effect of the electric force on the motion may be neglected and the free paths may be taken to be straight lines. [The electric force may be supposed to be in the direction of the axis z.]

The ordinate x is the sum of the projections on the axis of x of the free paths of the electrons, which are at distances x from the plane A after they have traversed the number of free paths c. Thus if $l_1, l_2 \ldots l_c$ be the lengths of the free paths $\theta_1, \theta_2, \ldots \theta_c$, the inclinations of the paths to the axis of x, the ordinate is

$$x=l_1\cos\theta_1+l_2\cos\theta_2+\ldots l_c\cos\theta_c. \quad . \quad . \quad (3)$$

The free paths are distributed about the mean free path l, and the mean square

$$(l_1^2+l_2^2+\ldots+l_c^2)/c$$

of a large number of paths is $2l^2$, so that the coefficient of diffusion K is $l\mathrm{U}/3$.

It is convenient to express the integral $\int_z^\infty n\,dx$ in the form

$$\mathrm{N}\pi^{-\frac{1}{2}}\int_\phi^\infty \epsilon^{-\phi^2}d\phi, \quad \text{where} \quad \phi = x\sqrt{3/4l^2c},$$

since the values of this integral for different values of ϕ are given in logarithmic tables.

It is sufficient for this investigation to consider the values N/10, N/100, and N/1000 of the integral which correspond to the values ·92, 1·65, and 2·2 of the lower limit ϕ or to the values $\cdot 92\times l\times\sqrt{4c/3}$, $1\cdot 65\times l\sqrt{4c/3}$, and $2\cdot 2\times l\sqrt{4c/3}$ of the distance x.

Thus, in equation (3), if c be the number of terms on the right of the equation the probability of the value of x/l being greater than $\cdot 92\times\sqrt{4c/3}$ is 1 : 10, the probability of it being greater than $1\cdot 65\times\sqrt{4c/3}$ is 1 : 100, and the probability of it being greater than $2\cdot 2\times\sqrt{4c/3}$ is 1 : 1000.

8. This method of investigation may also be adopted to find the distribution of the values of the sums S of the projections of lines of equal length l on the axis of x, all directions of the lines in space being equally probable.

The values of S/l are the values of the sums of the cosines

$$\cos\theta_1+\cos\theta_2+\ \ldots\ \cos\theta_c,$$

where θ_1, θ_2 .... θ_c are the inclinations of the lines to the axis of x.

In order to find the values of S/l the changes in the positions of electrons due to diffusion may be compared with the changes in the positions of particles that traverse paths of equal length l, all directions of the paths being equally probable. Under these conditions the coefficient of diffusion would be $l\mathrm{U}/6$, the time to be required to traverse a number of paths c would be lc/U, and $4\mathrm{K}t$ becomes $2l^2c/3$.

All the calculations are therefore the same as in the previous investigation, except that $2c/3$ is substituted for $4c/3$; also l may be taken to be the unit of length.

Thus the probability of the sum

$$\mathrm{S}=\cos\theta_1+\cos\theta_2\ \ldots\ \cos\theta_c \quad . \quad . \quad . \quad (4)$$

being greater than $\cdot 92\times\sqrt{2c/3}$ is 1 : 10, the probability of it being greater than $1\cdot 65\times\sqrt{2c/3}$ is 1 : 100, and the probability of it being greater than $2\cdot 2\times\sqrt{2c/3}$ is 1 : 1000.

It will be observed that the values of x/l given by formula (3), where the free paths are distributed about the mean, are greater by the factor $\sqrt{2}$ than the values S given by equation (4), where all the free paths are of the same length.

9. When the energy E of an electron is large compared with that of a molecule the changes of the energy of the electron in collisions with a molecule may be represented by a small constant loss λE in all collisions, together with large variable losses or gains, depending on the kinetic energy and direction of motion of the molecule. It may be supposed that the molecule is an elastic sphere of mass M and the kinetic energy $MV_0^2/2$ equal to the mean energy of agitation of a molecule of a gas at 15° C. Let m be the mass of the electron, $mU^2/2$ its kinetic energy, and $mU^2=kMV_0^2$. If the value of k be 100, the kinetic energy of the electron is 3·7 volts.

In a large proportion of the collisions the directions of motion of the electrons are inclined at angles of about 45° to the normal to the sphere at the point of impact. In order to simplify the investigation it may therefore be assumed that the component of the velocity of the electron in the direction of the normal is $U/\sqrt{2}$. All directions of motion of the sphere are equally probable, and the velocity of the sphere in the direction of the normal at the point of impact lies between the values $\pm V_0$. If v be the velocity of the sphere in this direction, the loss of energy of the electron in a collision is

$$\frac{2mE}{M}(1\pm\sqrt{2M/mk}\times v/V_0). \quad . \quad . \quad . \quad . \quad (5)$$

Thus the constant mean loss of energy λE for all collisions is $2m$E/M.

The second term in the bracket represents an additional loss of energy (v positive) when the sphere is receding from the electron, and a gain of energy (v minus) when the sphere is approaching the electron. This term diminishes as k increases, but it cannot be neglected even for values of k of the order of 100. The losses of energy corresponding to that value of k in the collisions

of an electron with an atom of helium are given by the formula

$$\lambda E(1 \pm 12{\cdot}3 \times v/V_0),$$

and if each electron in a current made a large number of collisions with atoms of the gas there would be large inequalities in the total amounts of energy lost by the different electrons.

10. These inequalities depend on the values of the sum S of the ratios v/V_0. The ratio v/V_0 is the cosine of the angle between a normal, and the direction of motion of the sphere, so that the values of S are distributed in the same manner as the sum of the cosines in equation (4). If the number of collisions c of each electron with atoms of helium be of the order of 100, the mean loss of energy λEc is small compared with E, and it is not necessary to consider the changes in E in order to find the distribution of the losses of energy when a large number N of electrons collide with atoms of the gas.

The distribution of the losses

$$\lambda E(c \pm 12{\cdot}3 \times S) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

depends on the number c, and the figures given in section 7 show that the number of electrons for which S is greater than $\cdot 75 \times \sqrt{c}$ is N/10, the number for which S is greater than $1{\cdot}35 \times \sqrt{c}$ is N/100, and the number for which S is greater than $1{\cdot}8\sqrt{c}$ is N/1000. Thus if c be 100, one-tenth of the total number of electrons lose amounts of energy greater than $192 \times \lambda.E$ and a similar number lose less than $8 \times \lambda E$, one-hundredth of the total number lose more than $266 \times \lambda E$, and a similar number gain more than $66 \times \lambda E$, one-thousandth of the total number lose more than $321 \times \lambda E$, and a similar number gain more than $121 \times \lambda E$.

With gases of higher atomic weight these inequalities are much larger. For mercury vapour the numerical coefficient of S is 87 instead of 12·3 in equation (6).

Not only are there large inequalities in the amounts of energy lost by different electrons, but an appreciable proportion gain energy in the collisions. These large inequalities are due to the fact that the molecules of the gas are in motion. If the temperature were greater than 15°, so that the mean velocity of the electrons

were V instead of V_0, the term $\sqrt{2M/mk}$ in equation (5) would be increased by the factor V/V_0, and there would be a corresponding increase in the inequalities of the energies of the electrons.

These results show that if experiments are designed to measure these losses of energy, and it is found that all the electrons lose the amount of energy λEc, the experiments do not prove that the collisions are the same as the collisions between elastic spheres *.

11. In the steady state of motion under a uniform electric force the electrons acquire a mean energy of agitation $\overline{E}$ depending on the ratio of the force Z to the pressure p of the gas. If l be the mean free path of the electrons the energy $\overline{E}$ is given by the formula

$$\overline{E}=Zel/\sqrt{2\cdot46\times\lambda},$$

e being the atomic charge. The mean velocity $\overline{W}$ of the electrons in the direction of the electric force is small compared with the mean velocity of agitation $\overline{U}$, the ratio $\overline{U}/\overline{W}$ being $\sqrt{2\cdot46/\lambda}$.

In helium † the ratio of the velocities has been found to be almost exactly 100 for values of $\overline{E}$ from ·25 volt to 4 volts. The mean free paths are therefore straight lines uniformly distributed in all directions except for the slight curvature in the direction of the electric force which gives rise to the velocity $\overline{W}$.

When the mean energy of the electrons is large compared with the mean energy of the molecules of the gas the inequalities in the energies of the electrons, due to the motion of agitation in the electric field, are much greater than the inequalities due to differences in the amounts of energy lost in collisions with atoms. An approximate estimate of the distribution of the energies of the electrons about the mean energy $\overline{E}$ is therefore obtained by assuming that the loss of energy is the same in all collisions where the energies of the electrons are the same. The actual distribution is wider than that calculated on this hypothesis.

In the following calculations it will be assumed that the loss of energy of an electron is λE, where λ is constant, in all collisions where the energy of the electron is E.

* Proc. Roy. Soc. A, cxx. p. 511 (1928).

† J. S. Townsend and V. A. Bailey, Phil. Mag. xlvi. p. 657 (Oct. 1923).

12. The changes in the positions of electrons in the direction of the electric force may be found by adding the velocity W to the motion due to diffusion.

If $\overline{U}$ be the velocity of an electron along a free path of length l_1, U_1 the component of $\overline{U}$ in the direction of the electric force Z, the time required to traverse the free path is $l_1/\overline{U}$, and in that time the electron moves the distance

$$(W \pm U_1) l_1/\overline{U}$$

in the direction z, and gains the amount of energy

$$(W \pm U_1) Z e l_1/\overline{U}.$$

Let $\overline{E}$ be the mean energy of agitation of electrons in the steady state of motion maintained by the force Z, so that the relation between $\overline{E}$ and Z is

$$\overline{E} = Zel/\sqrt{2{\cdot}46 \times \lambda}.$$

The velocity $\overline{U}$ is $\sqrt{2\overline{E}/m}$, and the ratio $U_1/\overline{U}$ is the cosine of the angle θ_1 between the free path l_1 and the axis of Z.

Let all the electrons N in a large group have the same energy $\overline{E}$ initially. After traversing the free paths l_1, l_2 . . . l_c the gains of energy of the electrons are given by the formula

$$\sqrt{2{\cdot}46 \times \lambda} \times \overline{E}(Wc/\overline{U} \pm L/l), \quad . \quad . \quad . \quad (7)$$

where

$$L = l_1 \cos \theta_1 + l_2 \cos \theta_2 + \ldots l_c \cos \theta_c, \quad . \quad . \quad (8)$$

the values of L being distributed in the same way as the values of x in equation (3).

In order to simplify the formula (6) it is necessary to find the velocity W in terms of $\overline{U}$. In this case W is the velocity of a group of electrons all with the same energy $\overline{E}$, and is less than the mean velocity $\overline{W}$ as found experimentally. The velocity $\overline{W}$ is the mean velocity of electrons in the steady motion where the energies are distributed about the mean energy $\overline{E}$.

In order to avoid a somewhat complicated calculation it may be assumed as a first approximation that W is the same as $\overline{W}$, as this does not introduce any error in the second term in the bracket in formula (7). With small values of c (less than 1000) the gains and losses of energy of the different electrons depend principally on this term, so that a sufficient degree of accuracy is

attained by taking an approximate value for W in the first term.

Since

$$\overline{W}/\overline{U}=\sqrt{\lambda}/\sqrt{2\cdot46},$$

formula (7) for the gains of energy thus becomes

$$\lambda\overline{E}(c\pm\sqrt{2\cdot46}/\lambda\times L/l). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The distribution of the gains of energy among a large number of electrons N, after each has traversed the same number of free paths c, depends on the distribution of the values of L.

It will be observed that the variable part in the above expression (9) for the gain in energy of an electron is ZeL, L being the sum of the projections of the free paths of an electron on the axis of z, which is independent of the velocity of the electron.

12. The gain of energy $\lambda\overline{E}c$ (formula (9)) in the free paths is cancelled by the loss of energy $\lambda\overline{E}c$ (formula (5)) in the collisions, provided the difference between the energy E of the electron and the mean energy $\overline{E}$ is small.

When the electron attains an energy E greater than $\overline{E}$ the velocity W in the direction of the electric force is less than $\overline{W}$ and the time required to traverse a free path l is less than $l/\overline{U}$, so that the mean gain of energy in a free path due to the velocity W is reduced from $\lambda\overline{E}$ to $\lambda\overline{E}^2/E$. Also the mean loss of energy in a collision is increased to λE.

There is therefore a total loss of energy due to these causes when an electron starts with the energy $\overline{E}$ and has a larger energy E at the end of several collisions. When this loss is small compared with the increase of energy $(E-\overline{E})$ it is given approximately by the term

$$\lambda c(E^2-\overline{E}^2)/2E.$$

Thus under the combined effects of the gains of energy in moving along free paths and the losses of energy in collisions the changes in the energies of the electrons are given by the formula

$$E-\overline{E}=\lambda\overline{E}[c(E^2-\overline{E}^2)/2E\overline{E}\pm\sqrt{2\cdot46}/\lambda L/l] \quad . \quad (10)$$

when the first term in the bracket is small compared with the second.

This formula is therefore convenient for calculating increases in the energies when the number c is not very large (less than 2000). In helium the coefficient $\sqrt{2\cdot46/\lambda}$ is approximately 100, which is much greater than the corresponding coefficient 12·3 in equation (6).

14. The distribution of the gains in energy among a large number of electrons when each electron traverses the same number of free paths c depends on the values of L in equation (10), which are the same as the values of x in equation (3).

Thus for one-tenth of the total number N of electrons the lengths L are greater than $1\cdot06\times l\sqrt{c}$, for one-hundredth of the total number the lengths L are greater than $1\cdot9l\sqrt{c}$, and for one-thousandth of the total number the lengths L are greater than $2\cdot53l\sqrt{c}$.

It is of interest to indicate the rate at which the energies E of some of the electrons increase when a large number N move in helium under the action of an electric force, all the electrons having the same energy $\bar{E}$ initially. The number that have energies greater than a certain value may be found by equation (10). Thus after the electrons have traversed 100 free paths, ten per cent. of the total number have energies greater than $1\cdot26\times\bar{E}$, one per cent. have energies greater than $1\cdot46\times\bar{E}$, and one in a thousand have energies greater than $1\cdot61\times E$. After larger numbers of collisions the energies are more widely distributed, as is shown by the figures in the following table representing the distributions when the electrons have traversed 100, 400, and 900 collisions. The table gives the values of $E/\bar{E}$, which are exceeded by the numbers N/10, N/100, and N/1000.

C.	N/10.	N/100.	N/1000.
100	1·26	1·46	1·61
400	1·48	1·87	2·17
900	1·66	2·12	2·60

Thus after 900 collisions one per cent. of the total number of electrons have energies greater than $2\cdot12\times\bar{E}$ and one in a thousand have energies greater than $2\cdot62\times\bar{E}$.

15. The calculations show that if the energies of any group of electrons be approximately equal to the mean energy at any time they do not remain at that value, but they rapidly acquire different energies above and below the mean energy.

The above investigation is a simple application of the method of finding the distributions of the energies which has already been described in a previous paper *. The final distribution of the energies after the electrons have traversed a large number of free paths may be found by the method explained in sections 22–29 of that paper. This distribution for energies greater than the mean is given approximately by the formula

$$dn = \mathrm{N}.\pi^{-\frac{1}{2}}\epsilon^{-(\mathrm{E}/\mathrm{E}_1-1)^2}d\mathrm{E}/\bar{\mathrm{E}}, \quad . \quad . \quad . \quad (11)$$

N being the total number and dn the number with energies between E and (E+dE). As already explained, this equation does not take into consideration the inequalities in the energies due to the differences in the losses of energy in collisions with molecules of the gas as given by equation (9). The actual distribution is therefore somewhat wider than that indicated by equation (11).

16. In his remarks on that investigation Didlaukis states that there is an error in the result because changes in the energy of the electrons are not considered in certain differential equations representing the process of diffusion.

He appears to have overlooked the fact that projections of free paths on any axis may be deduced from equations representing diffusion of particles in which the energy of the particles is constant. Sections 18–21 of the first paper (June 1930) deals with the projections of free paths when there is no electric force acting, and the energy of the electrons may be supposed to be either constant or variable provided the velocities of the electron are large compared with the velocities of the molecules of the gas, as in section 6 of this paper.

The changes in the energies of the electrons moving in an electric field are correctly determined in sections 22 and 23 of the first paper (June 1930), the principle being the same as in sections 12 and 13 of this paper, where there are no differential equations.

* Phil. Mag. ix. p. 1145 (June 1930).

17. The results of the experiments on the diffusion of electrons in helium provide a simple means of testing the validity of the conclusion arrived at by Didlaukis, to the effect that in the steady motion under an electric force all the electrons have practically the same energy $\overline{E}$, assuming that in each collision with an atom of the gas the loss of energy is a constant λE.

If these hypotheses be accepted the electrons would all move with the same velocity $\overline{U}$ along the free paths, and if they were at a plane A initially their distances z from the plane due to diffusion, after each electron had traversed the same number of free paths c, is given by the formula $n=n_0\epsilon^{-3z^2/4l^2c}$ ($n\,dz$ being the number at distances between z and $z+dz$).

In addition there would be a uniform velocity $W=U/100$ in the direction of the electric force and a uniform displacement $a=lc/100$ after each electron had traversed the free paths. Taking c to be 4000, the distance a is $40\times l$, and the distribution due to diffusion is given by the formula $n=n_0\times\epsilon^{-3z^2/10a^2}$.

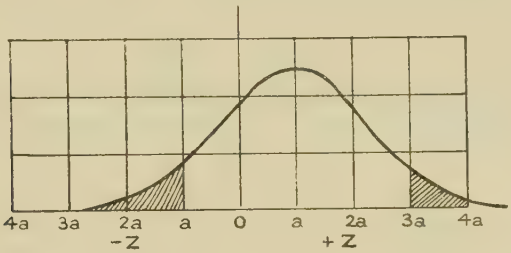
Under these conditions the space distribution of the electrons after each has traversed 4000 free paths is given by the curve (see figure), where the concentration at the distance z from the plane A($z=0$) is proportional to the ordinate, and the distance z is given by the abscissa in terms of the distance a.

Also, by hypotheses, the electrons have the same energy throughout the motion and each has lost the amount of energy 4000 $\lambda\overline{E}=\overline{E}$ in collisions with atoms of the gas.

But according to the ordinary laws of electrostatics an electron in traversing the distance a in the direction of the electric force acquires the amount of energy Zea which in this case is $Zel/\sqrt{2\cdot46}\lambda=E$. Thus the electrons which traverse the distance $3a$ in the direction of the electric force gain the amount of energy 3E, and those that traverse the distance a in the opposite direction lose the amount of energy $\overline{E}$.

These hypotheses therefore lead to the conclusion that the number of electrons represented by the shaded area on the right of the figure have energies greater than $3\overline{E}$, and a similar number represented by the shaded area on the left have energies less than $-E$.

It therefore appears that the laws of collisions, which include the hypothesis that in the steady state of motion all the energies of the electrons are approximately the same, are not in accordance with ordinary mechanics or the results of simple experiments on diffusion.



18. The preceding calculations show that in the steady motion of electrons moving under the action of a uniform electric force Z there is a wide distribution of the energies about the mean energy E, and the velocities $\bar{W}$ and $\bar{U}$ found experimentally * are the mean values of these quantities for all the electrons in the current.

There are also wide variations in the velocities of the electrons in the direction of the electric force, and the velocity W_1 of those electrons which have the energy of agitation $\bar{E}$ is less than $\bar{W}$.

The formula which is generally given for the velocity W_1 in terms of $\bar{U}$ is $Zel/m\bar{U}$, but this is only an approximate formula. A more accurate formula for the velocity W_1 of a group of electrons all with the same velocity of agitation $\bar{U}$ is $2Zel/3m\bar{U}$.

The mean velocity $\bar{W}$ of all the electrons in a current, when the motion is steady depends on the distribution of the energies of the electrons. The formula deduced from Pidduck's investigation is $\bar{W}=\cdot 91\times Zel/m\bar{U}$, which gives $W_1=\cdot 73\times\bar{W}$. It would then be more accurate to take this value for W in equation (7).

This would have very little effect on the calculation of the energies for values of c less than 400, but with larger values of c the number of electrons with the larger energies would be diminished. Thus when $c=900$

* 'Motion of Electrons in Gases' (Clarendon Press, Oxford).

the values of $E/\overline{E}$ corresponding to the values of N/10, N/100, and N/1000 would be 1·5, 2·0, and 2·4 respectively instead of the numbers given in the last line in the table.

Thus the distribution of the energies depends to some extent on the ratio $W_1/\overline{W}$. This point will be discussed more fully in another publication.

LXV. *The Mathematical Transformation of the Set-square-index Nomogram.* *By* H. J. Allcock, *M.Sc., A.M.I.E.E., A.M.I.Mech.E.**

Contents.

1. Introduction.

ALTHOUGH the nomogram provides a useful aid to computation, its present employment is generally confined to those types in which the required alignment is carried out by means of an index consisting of a straight line. The set-square-index nomogram, the index of which consists of two perpendicular straight lines, has been somewhat neglected owing to the incompleteness, in technical literature, of the theory underlying its construction. In particular the problems of the use of scale factors and of transformation have received little attention.

Although the theory of the set-square-index nomogram has been described elsewhere [1]†, it will be convenient to give here a recapitulation. The determinant,

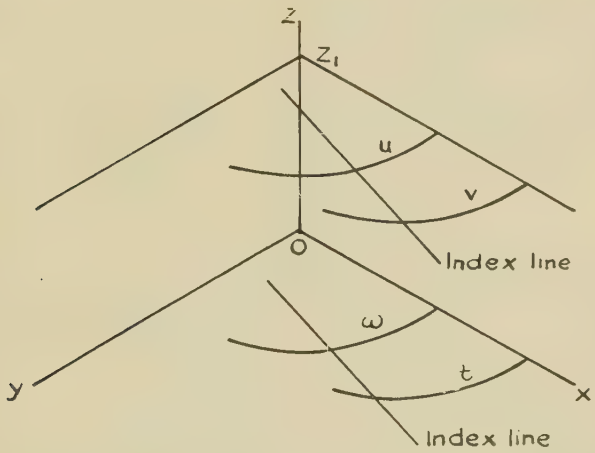
$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_1 & 1 \\ x_3 & y_3 & 0 & 1 \\ x_4 & y_4 & 0 & 1 \end{vmatrix} = 0, \quad . \quad . \quad . \quad (1)$$

* Communicated by Prof. G. H. Livens, B.A.

† See Bibliography, p. 758.

expresses the condition that the two points (x_1, y_1, z_1) and (x_2, y_2, z_1) lying in the plane $z=z_1$, and the two points $(x_3, y_3, 0)$ and $(x_4, y_4, 0)$, lying in the plane $z=0$, shall lie also in one common plane, as shown in fig. 1. Since the two planes, $z=z_1$ and $z=0$, are parallel, their lines of intersection with the plane common to the four points will also be parallel. If, therefore, in the case of a formula containing four variables it is possible to form a determinant similar to (1), in which each row contains functions of one variable only, the third column consists of equal constants and zero only, and the fourth

Fig. 1.



column contains unity only, then a nomogram may be drawn on two parallel planes, solutions of the formula being obtained by the use of an index consisting of two parallel straight lines, one situated in each plane. It is of course impracticable to deal with such a nomogram drawn in the solid ; and a more useful form may readily be obtained by superimposing the plane $z=z_1$ directly on the plane $z=0$. The same conditions will still hold, and the nomogram may still be read by means of two parallel straight lines, the first cutting the loci contained originally in the plane $z=z_1$, and the second the loci contained originally in the plane $z=0$. This form is known as the parallel-index nomogram, and is illustrated in fig. 2 *a*.

If after superposition the plane $z=z_1$ is turned through 90° about a normal to the plane, the two index lines, which have hitherto been parallel, will now intersect at right angles. The nomogram may now be read

Fig. 2 *a*.

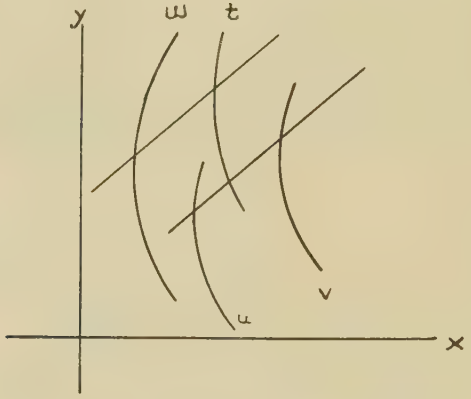
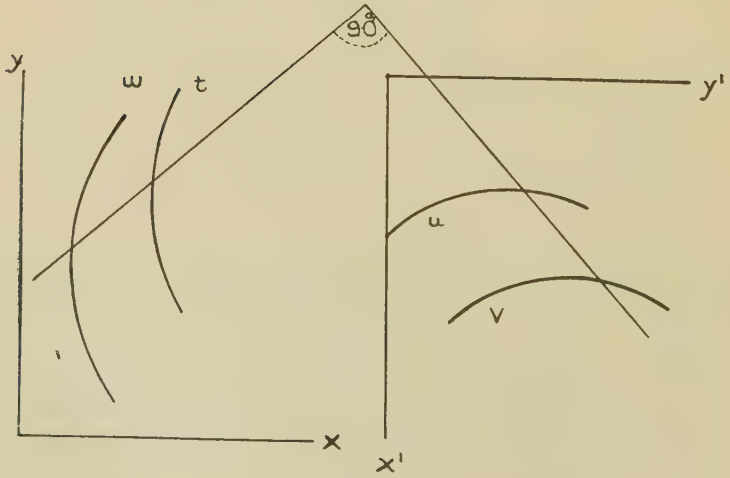


Fig. 2 *b*.



by means of two perpendicular straight lines, such as the edges of a set-square. This form is known as the set-square-index nomogram, and is illustrated in fig 2 *b*.

It will thus be seen that the two forms of nomogram are defined by the same determinant. In practice

the set-square-index nomogram will be found more convenient to use, as one pair of loci is drawn clear of the other pair; but the theory underlying the construction of either form is the same. In either case the value of z_1 is immaterial (subject to the sole condition that it must not be equal to zero) and may be taken as unity, so that the general basic determinant for either form of nomogram is

$$\begin{vmatrix} g(u) & f(u) & 1 & 1 \\ g(v) & f(v) & 1 & 1 \\ g(w) & f(w) & 0 & 1 \\ g(t) & f(t) & 0 & 1 \end{vmatrix} = 0. \quad . \quad . \quad (2)$$

The purpose of this thesis is to show that an inherent limitation of construction exists for this type of nomogram. No attempt has yet been made to effect a mathematical transformation. The theory underlying such a transformation is therefore examined, and it is shown that a suitable transformation is possible only if certain conditions are fulfilled.

2. Historical Survey.

The parallel-index nomogram was first described by Beghin [2] in 1892. He pointed out that a nomogram may be constructed for the formula

$$\frac{g(u)-g(v)}{f(u)-f(v)} = \frac{g(w)-g(t)}{f(w)-f(t)}$$

by drawing the four loci of u, v, w, and t (by means of the coordinates $x_1=g(u_1)$, $y_1=f(u_1)$, etc.), which may be read by means of two parallel index lines, the first cutting the loci of u and v and the second the loci of w and t. Certain scale factors are employed, reference to which is made later *.

In 1898 Goedseels [3] pointed out the convenience of using two perpendicular straight lines as an index. Using the same general method as Beghin he constructed a modified form of nomogram, with one pair of loci turned through 90° relative to the other pair.

The first use of the complete determinant for the two forms was made by Soreau [4] in 1901. He introduced the complete determinant for the double-alignment

* See *post*, p. 750.

nomogram, and showed that it could be transformed to a form similar to (2) by multiplying by another determinant of the fourth order of suitable form, not equal to zero. Further, he commented on the fact that the parallel-index nomogram could be considered as a double-alignment nomogram with a reference line lying at infinity. In 1906, however, he [5] showed that, instead of using a multiplying determinant, the more obvious method which follows could be employed.

The complete basic determinant for the double-Alignment Nomogram is

$$\begin{vmatrix} g(u) & f(u) & 0 & 1 \\ g(v) & f(v) & 0 & 1 \\ g(w) & 0 & f(w) & 1 \\ g(t) & 0 & f(t) & 1 \end{vmatrix} = 0. \quad . \quad . \quad (3)$$

If the third column is replaced by the sum of the second and third columns, and each row is then divided by the new third element so obtained, and the columns are rearranged in the order 1, 4, 2, 3, then the following determinant

$$\begin{vmatrix} \frac{g(u)}{f(u)} & \frac{1}{f(u)} & 1 & 1 \\ \frac{g(v)}{f(v)} & \frac{1}{f(v)} & 1 & 1 \\ \frac{g(w)}{f(w)} & \frac{1}{f(w)} & 0 & 1 \\ \frac{g(t)}{f(t)} & \frac{1}{f(t)} & 0 & 1 \end{vmatrix} = 0, \quad . \quad (4)$$

which is of the same form as (2), is obtained.

In 1921, in the second edition of his 'Traité de Nomographie,' d'Ocagne [6] indicated that, by employing Soreau's complete determinant and a suitable multiplying determinant of sixteen parameters, a mathematical transformation of a double-alignment nomogram could be effected. In passing it should be pointed out that, although d'Ocagne makes no mention of the fact, it is essential for six of the sixteen elements of the multiplying determinant to be equal to zero, since otherwise the form of determinant typical of double alignment will

be lost. An example of the use of this method of transformation has been given elsewhere[1].

In more recent publications several authors, notably von Dobbeler[7], have described the construction of both the parallel-index and the set-square-index nomograms; but in every case the problem has been treated as one in pure geometry.

In the case of nomograms of single alignment both Soreau[4] and d'Ocagne have shown, by theory and example, that suitable mathematical transformations can be carried out: in the case of double-alignment nomograms they have both indicated, without example, a method of mathematical transformation; but in no one of the references cited is mention made as to whether a mathematical transformation is possible in the case of the set-square-index or parallel-index nomograms.

3. Scale Factors.

It is sometimes possible by the use of oblique axes to rearrange the positions of loci relative to one another, so that the resulting nomogram may be drawn in the most convenient practical form. In other cases, however, no suitable obliquity can be found, and recourse must be had to a modification of the scale factors in order to alter the lengths of the loci (between their required limits) relative to one another. This may be done in the case of nomograms of the third and fourth classes, with some few exceptions including the set-square-index nomogram. In this last type the only possible scale factors which may be introduced into the basic determinant (2) are the same for (say) both x elements of the loci in one half of the nomogram. It will thus be seen that the relative lengths of a pair of loci cannot be changed. Even if the basic determinant be transformed to that for double alignment, the appropriate scale factors introduced, and the modified determinant retransformed, it will be found that the same scale factors as before are obtained. The general constructional determinant must therefore be of the form

$$\begin{vmatrix} \delta_1 g(u) & \mu_1 f(u) & 1 & 1 \\ \delta_1 g(v) & \mu_1 f(v) & 1 & 1 \\ k\delta_1 g(w) & k\mu_1 f(w) & 0 & 1 \\ k\delta_1 g(t) & k\mu_1 f(t) & 0 & 1 \end{vmatrix} = 0. \quad . \quad (5)$$

The scale factors appearing in this determinant are identical with those used by both Beghin * and Goedseels.

4. Mathematical Transformation.

4 *a*. *General.*

In practice cases will occur where, owing to the type of formula under consideration, neither the use of oblique axes nor modifications of the scale factors will provide a nomogram of convenient practical form. For example, in the nomogram of the third class, illustrated in

Fig. 3 *a*.

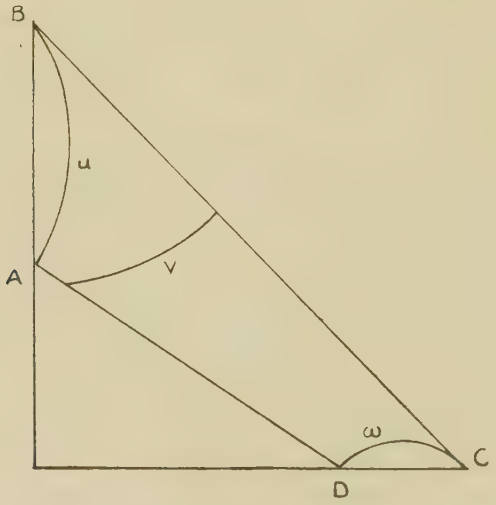


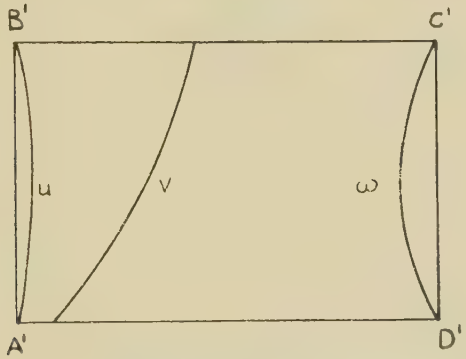
fig. 3 *a*, the loci are of unequal lengths and appear at somewhat awkward angles to one another. A more useful nomogram would be that shown in fig. 3 *b*, where the loci are of more or less equal lengths and appear "opposite" one another. By the methods of conical projection the one nomogram may, in fact, be transformed graphically to the other, since the limiting figure (*i. e.*, the figure obtained by joining the four points forming the lower and upper limits of the two boundary loci) in the case of fig. 3 *a* is the quadrilateral ABCD and in the case of fig. 3 *b* is the rectangle A′B′C′D′. If, however, the original basic determinant can be modified

* See *ante*, p. 747.

without alteration of the formula, the transformation may be carried out mathematically.

If the basic determinant be multiplied by another determinant of the same order, not equal to zero and containing parameters only, the formula will remain unaltered, but the parameters will be introduced into the determinant without changing its characteristic form, in which each row contains functions of only one variable and the last column unity only. A sufficient number of simultaneous equations may be obtained from this modified determinant by substituting the lower and upper limiting values of the variables (whose loci form the boundaries of the nomogram) in their respective x and y elements, and equating the resulting expressions

Fig. 3 *b*.



to values of x and y such that the limits of the loci will lie at the corners of a rectangle. From these equations the relationship existing between the parameters may be found and, by arbitrarily choosing a numerical value for one of them (not equal to zero), the numerical values of the others can be determined. If these values be substituted in the modified determinant the resulting nomogram will be such that its limiting figure is a rectangle, and this rectangle may be made of any required size by multiplying the first and second columns of the determinant by an appropriate numerical constant in each case.

4 *b*. *The Double-alignment Nomogram.*

A similar mathematical transformation can be effected in the case of the double-alignment nomogram, the

limiting figure after transformation being a triangular prism, as shown in fig. 4. The multiplying determinant of the fourth order must have parameters so arranged that the form of determinant peculiar to double-alignment is retained after multiplication, and must therefore be of the form

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix} = D \neq 0 \quad . \quad (6)$$

The determinant, resulting from multiplying (3) by (6), will be of the form

$$\begin{vmatrix} \alpha_1 g(u)+\beta_1 f(u)+\delta_1 & \beta_2 f(u) & 0 & 1 \\ \alpha_4 g(u)+\beta_4 f(u)+\delta_4 & \alpha_4 g(u)+\beta_4 f(u)+\delta_4 & & \\ \alpha_1 g(v)+\beta_1 f(v)+\delta_1 & \beta_2 f(v) & 0 & 1 \\ \alpha_4 g(v)+\beta_4 f(v)+\delta_4 & \alpha_4 g(v)+\beta_4 f(v)+\delta_4 & & \\ \alpha_1 g(w)+\gamma_1 f(w)+\delta_1 & 0 & \gamma_3 f(w) & 1 \\ \alpha_4 g(w)+\gamma_4 f(w)+\delta_4 & & \alpha_4 g(w)+\gamma_4 f(w)+\delta_4 & \\ \alpha_1 g(t)+\gamma_1 f(t)+\delta_1 & 0 & \gamma_3 f(t) & 1 \\ \alpha_4 g(t)+\gamma_4 f(t)+\delta_4 & & \alpha_4 g(t)+\gamma_4 f(t)+\delta_4 & \end{vmatrix} = 0. \quad . \quad (7)$$

Eight simultaneous equations may now be obtained, as before, by considering the respective positions of the lower and upper limiting graduations of the loci, u and t, which form the upper boundaries of the nomogram. Since the planes, ADE and BCF, must be parallel (fig. 4), two further equations may be obtained by substituting the lower and upper limiting values of a third variable (say, v) in its x element, and equating the resulting expressions to zero and x_2 respectively. From these ten simultaneous equations the numerical values of the ten parameters may be determined and substituted in their appropriate positions in (7).

4 *c. The set-square-index nomogram.*

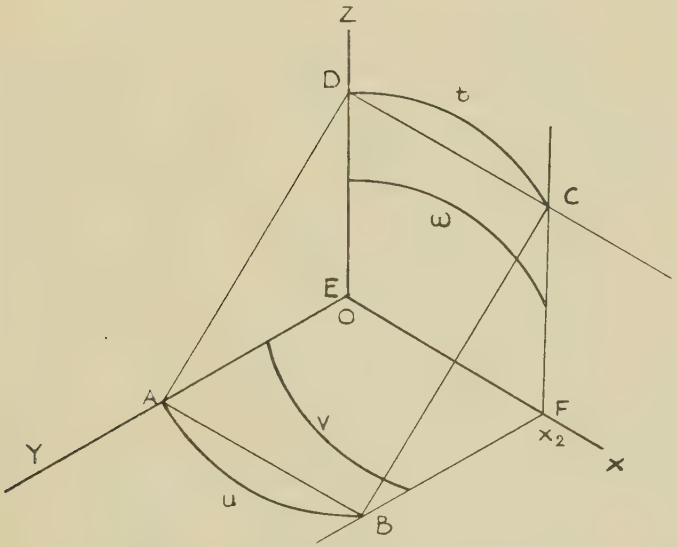
There are three methods by which parameters may be introduced into the basic determinant (2) for the

set-square-index nomogram while still retaining the characteristic form. They are as follows :—

a. The basic determinant (2) for the set-square-index nomogram may be multiplied by the determinant :

$$\begin{vmatrix} \delta_1 & \alpha_1 & \beta_1 & \gamma_1 \\ \delta_4 & \alpha_4 & \beta_4 & \gamma_4 \\ 0 & 0 & \beta_2 & 0 \\ 0 & 0 & 0 & \gamma_3 \end{vmatrix} = D \neq 0.$$

Fig. 4.



b. The basic determinant (3) for the double-alignment nomogram may be multiplied by the determinant :

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ 0 & \beta_2 & \gamma_3 & 0 \\ 0 & \beta_2 & 0 & 0 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix} = D \neq 0.$$

c. The transformed determinant (7) for the double-alignment nomogram may be further modified by replacing the third column by the sum of the second and third, dividing each row by the new third element so obtained, and rearranging the columns in the order 1, 4, 2, 3.

In each case, the result will be the same; a determinant will finally be obtained of the form:

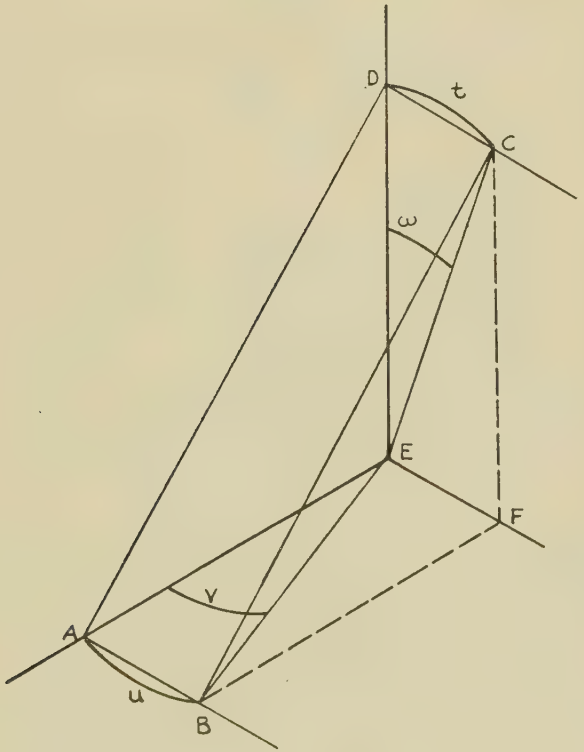
$$\begin{vmatrix} \dfrac{\alpha_1 f(u)+\beta_1+\delta_1 g(u)}{\beta_2} & \dfrac{\alpha_4 f(u)+\beta_4+\delta_4 g(u)}{\beta_2} & 1 & 1 \\ \dfrac{\alpha_1 f(v)+\beta_1+\delta_1 g(v)}{\beta_2} & \dfrac{\alpha_4 f(v)+\beta_4+\delta_4 g(v)}{\beta_2} & 1 & 1 \\ \dfrac{\alpha_1 f(w)+\gamma_1+\delta_1 g(w)}{\gamma_3} & \dfrac{\alpha_4 f(w)+\gamma_4+\delta\ g(w)}{\gamma_3} & 0 & 1 \\ \dfrac{\alpha_1 f(t)+\gamma_1+\delta_1 g(t)}{\gamma_3} & \dfrac{\alpha_4 f(t)+\gamma_4+\delta_4 g(t)}{\gamma_3} & 0 & 1 \end{vmatrix} = 0. \quad (8)$$

This is therefore the determinant from which the necessary simultaneous equations must be obtained; but any attempt to obtain the numerical values of the parameters by making use of such values of the coordinates that the two halves of the final nomogram are rectangular in shape will end in failure except (as will be shown later) in the case of certain definite combinations of limiting values of the variables in the formula under consideration. In view of this fact an endeavour was made to use a method of approximation, an example of which is given in the Appendix. The method consisted of making a number of transformations of the same determinant to give double-alignment nomograms, using the same limiting values for three of the variables in each case, except for the upper limit of one, where a range of values was used. Each of these nomograms was then transformed into a set-square-index nomogram, in order to determine whether in any one of them the required value of the upper limit of the one variable might finally appear in the appropriate position. This was, in effect, a use of method *c* above, but with definite numerical values substituted for the parameters before final transformation. It was found that the upper limiting value of the variable obtained after transformation was the same in every case, irrespective of the limiting values originally chosen before transformation.

A reason is now apparent for the failure to effect a suitable mathematical transformation. The set-square-index nomogram may be regarded as a double-alignment nomogram, the reference-line of which has been projected to infinity; and this is so even if the nomogram be

constructed from a transformed determinant containing parameters. This being so, all lines which in the double-alignment nomogram intersect on the reference-line will, in the set-square-index nomogram, become parallel. Therefore the parallel lines forming the lower and upper boundaries of the set-square-index nomogram (*i. e.*, the lines joining the lower and upper limiting

Fig. 5.



graduations respectively of the loci in each half of the nomogram) must, in the corresponding double-alignment nomogram, all intersect on the reference-line, as shown in fig. 5. Thus, whereas the limiting figure of the double-alignment nomogram is the triangular prism ABCDEF (fig. 4), the figure which corresponds (before transformation) to the limiting figure (after transformation) of the set-square-index nomogram is the rectangular pyramid ABCDE (fig. 5), with the apex situated on the reference-line.

The formula represented by a double-alignment or a set-square-index nomogram may be divided into two equal parts, being then of the form

$$f(u, v)=\mathrm{R}=f(w, t),$$

the loci of u and v appearing in one half of the nomogram and the loci of w and t in the other half, the reference-line being represented by R. It is obvious that, although not usually marked on the reference-line there is one definite value of R for any given pair of values of u and v, the position of that value being at the point at which the reference-line is cut by the line drawn through the given values on the loci of u and v. If therefore two lines passing through two pairs of graduations of u and v intersect on the reference line, the two values of the half of the formula containing u and v, obtained by substituting therein the respective numerical values of u and v, must be equal. It therefore follows that there is a definite relationship connecting the lower and upper limiting values of the two variables in one half of the set-square-index nomogram, which has been transformed to rectangular shape, since in the corresponding double-alignment nomogram the lines passing through the graduations representing these values intersect on the reference-line. Thus :

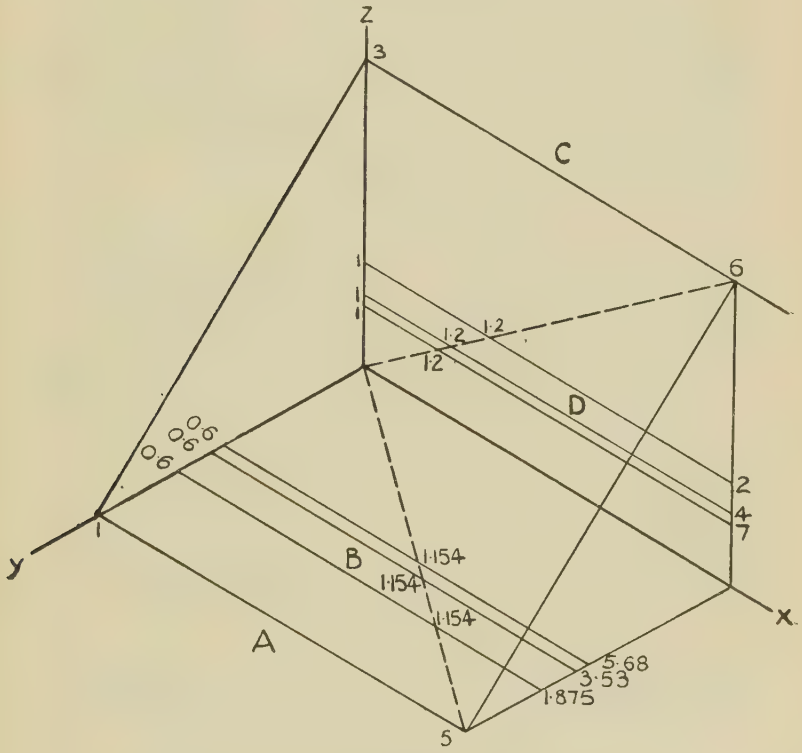
$$f(u_1, v_1)=\mathrm{R}_1=f(u_2, v_2), \quad . \quad . \quad . \quad . \quad (9)$$

from which it will be seen that if the lower and upper limiting values of u and the lower limiting value of v are fixed the upper limiting value of v must also be fixed. It will thus be seen that it is impossible to obtain a transformed set-square-index nomogram, the two halves of which will be rectangular, unless (9) is satisfied. Given three limiting values (for two variables in the one half of a formula) the appropriate fourth limiting value may be immediately calculated from (9) ; and if this value is not very different from that actually required a transformation may usefully be made, although the portion of the locus lying between the calculated limiting value and the required limiting value will appear outside the rectangle.

If the limiting values in the formula are such that a rectangular transformation of the set-square-index nomogram is possible, then it will be obvious, from a consideration of fig. 5, that a rectangular transformation

in the case of a double-alignment nomogram for the same formula and for the same limiting values will be impossible. The two types of nomogram may thus be considered as complementary, and where a transformation of the one type cannot be made for a given set of limits the other type may be used. In any case it will be possible to determine from (9) which type of nomogram can be suitably transformed.

Fig. 6.



5. Conclusions.

The set-square-index nomogram (and its associated form the parallel-index nomogram) may be regarded as a double-alignment nomogram with a reference-line situated at infinity. It follows that the parallel lines defining the limits of the rectangular set-square-index nomogram will, in the corresponding double-alignment nomogram before final transformation, inter-

sect on the reference-line. This being so, a relationship exists between the four limiting values of the two variables in any one half of the set-square-index nomogram, such that, three of them being chosen, the fourth is definitely fixed. For any given set of limiting values therefore it may not be possible for a rectangular transformation to be made.

The fact that a rectangular transformation between given limits is not always possible is a serious limitation to the practical utility of this type of nomogram, resulting, it may be, in an inconvenient figure and consequent inaccuracies in use. Unless therefore a formula, and the limiting values of its variables, are peculiarly adapted to illustration by this type, use should preferably be made of some other type, such as the double-alignment nomogram, or the grid nomogram.

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Appendix.

The Transformation of a Set-square-index Nomogram.

Consider the formula

$$\frac{1}{A}-\frac{1}{B}=\frac{1}{C}-\frac{1}{D}, \quad \ldots\ldots \quad (a)$$

which may be expressed in the form of a determinant typical of the double-alignment nomogram as

$$\begin{vmatrix} 0 & A & 0 & 1 \\ B & B & 0 & 1 \\ 0 & 0 & C & 1 \\ D & 0 & D & 1 \end{vmatrix} = 0. \quad \ldots \quad (b)$$

Multiplying this by (6), the following determinant is obtained :

$$\begin{vmatrix} \frac{\beta_1 A+\delta_1}{\beta_4 A+\delta_4} & \frac{\beta_2 A}{\beta_4 A+\delta_4} & 0 & 1 \\ \frac{\alpha_1 B+\beta_1 B+\delta_1}{\alpha B+\beta_4 B+\delta_4} & \frac{\beta_2 B}{\alpha_4 B+\beta_4 B+\delta_4} & 0 & 1 \\ \frac{\gamma_1 C+\delta_1}{\gamma_4 C+\delta_4} & 0 & \frac{\gamma_3 C}{\gamma_4 C+\delta_4} & 1 \\ \frac{\alpha_1 D+\gamma_1 D+\delta_1}{\alpha_4 D+\gamma_4 D+\delta_4} & 0 & \frac{\gamma_3 D}{\alpha_4 D+\gamma_4 D+\delta_4} & 1 \end{vmatrix} = 0. \quad (c)$$

If the required limiting values of A and C are 1 and 5, and 3 and 6, respectively, and the lower limiting value of D is 1, then, assuming certain upper limiting values of D, the necessary simultaneous equations for rectangular transformation of the double-alignment nomogram may be obtained and solved.

Taking in succession the values D=2, D=4, and D=7, solving the necessary simultaneous equations and substituting the numerical values of the parameters so obtained in (*c*), the following determinants are obtained :—

(D=2)
$$\begin{vmatrix} \frac{5A-5}{4A} & 1 & 0 & 1 \\ \frac{25B-15}{17B} & \frac{12}{17} & 0 & 1 \\ \frac{2C-6}{C} & 0 & 1 & 1 \\ \frac{2D-2}{D} & 0 & \frac{1}{3} & 1 \end{vmatrix} = 0. \quad (d)$$

(D=4)
$$\begin{vmatrix} \frac{5A-5}{4} & 1 & 0 & 1 \\ \frac{100B-60}{83B} & \frac{48}{83} & 0 & 1 \\ \frac{2C-6}{C} & 0 & 1 & 1 \\ \frac{4D-4}{3D} & 0 & \frac{2}{9} & 1 \end{vmatrix} = 0. \quad (e)$$

$$(D=7)\quad \begin{vmatrix} \frac{5A-5}{4A} & 1 & 0 & 1 \\ \frac{350B-210}{313B} & \frac{168}{313} & 0 & 1 \\ \frac{2C-6}{C} & 0 & 1 & 1 \\ \frac{7D-7}{6D} & 0 & \frac{7}{36} & 1 \end{vmatrix} = 0. \quad . \quad (f)$$

The composite nomogram obtained from these determinants is illustrated in fig. 6. The lines joining the origin to the upper limits of the loci of A and C respectively (*i. e.*, the lines which, after final transformation, would form the upper boundaries of the limiting rectangles of the set-square-index nomogram) being drawn, their points of intersection with the various loci of B and D may be calculated from the respective determinants. In each case the values of B and D will be found to be $B=1{\cdot}154$ and $D=1{\cdot}2$, and these are the limiting values which will be obtained in the set-square-index nomogram after final transformation.

The same limiting values are obtained by the use of (9), for

$$\frac{1}{A_1}-\frac{1}{B_1}=\frac{1}{A_2}-\frac{1}{B_2}=R_1,$$

or

$$\frac{1}{1}-\frac{1}{0{\cdot}6}=\frac{1}{5}-\frac{1}{B_2},$$

whence

$$B_2=1{\cdot}154\ ;$$

and

$$\frac{1}{C_1}-\frac{1}{D_1}=\frac{1}{C_2}-\frac{1}{D_2}=R_1,$$

or

$$\frac{1}{3}-\frac{1}{1}=\frac{1}{6}-\frac{1}{D_2}$$

whence

$$D_2=1{\cdot}2.$$

London.
November 1932.

LXVI. *Inner Potential of Metals.*

By James A. Darbyshire *.

[Plate XIX.]

Introduction.

SEVERAL attempts have been made to measure inner potentials by examining the diffraction effects obtained with slow and fast electrons. The inner potential appears in the form of a refractive index $\mu=\sqrt{1+\frac{E_0}{V}}$ †, where E_0 is the inner potential and V the accelerating voltage applied to the incident electrons. It was the discrepancy between the positions of the diffracted beams as predicted by the ordinary Bragg formula and the observed positions ‡ which first led to the introduction of an effective refractive index, and which was shortly afterwards correlated with the inner potential by Bethe †.

From the formula for μ it is apparent that the refractive index becomes large for small values of the incident electron velocity. Hence the actual displacement of the reflected beam becomes more appreciable when using low voltage electrons.

For high energy electrons, however, the refractive index is very small. Taking a typical case, for $E_0=15$ volts and $V=30{,}000$ volts we have $\mu=1{\cdot}00025$, whereas for $V=100$, $\mu=1{\cdot}07$. However, the discrepancy to be expected from the ordinary Bragg formula is still quite appreciable for very small glancing angles of reflexion §. This occurs because the deviation due to refractive index of a beam entering a plane surface is greatest under these conditions, and if we consider the reflexion from a set of planes of spacing d parallel to the surface, the condition for reinforcement is $n\lambda'=2d\sin\theta'$, where λ' and θ' represent the wave-length and glancing angle inside the crystal. The emergent beam makes the same angle with the surface as the

* Communicated by Prof. G. P. Thomson, M.A.

† H. Bethe, Naturwissenschaften, xv. p. 787 (1927); and xvi. p. 333 (1927).

‡ C. J. Davisson and L. H. Germer, Phys. Rev. xxx. p. 705 (1927).

§ G. P. Thomson, 'Wave Mechanics,' p. 95 (1930).

incident beam whatever the value of the refractive index. If μ is unity, the condition for reflexion is that the planes must be set at an angle θ to the incident beam given by $n\lambda=2d\sin\theta$, where λ is the wave-length *in vacuo* and also, in this case, in the crystal. If μ is greater than unity the condition for reinforcement from these planes is $n\lambda'=2d\sin\theta'$, and this will be satisfied inside the crystal for a smaller value of the external glancing angle θ by virtue of the deviation on entry already mentioned. But in all cases the observed deflexion of the beam is 2θ, therefore the reflected beam is deflected less in this case than in the case where μ is unity. The condition for reflexion for a beam of wave-length λ falling on a surface at glancing angle θ is

$$\sin\theta=\frac{n\lambda}{2d}\sqrt{1-\frac{8E_0med^2}{n^2h^2}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(E_0 in E.S.U.),

where h is Planck's constant, e and m the charge and mass of the electron respectively, $\frac{d}{n}$ the plane spacing divided by the order of reflexion. If s denotes the distance of a diffracted spot from the point of incidence of the direct beam on the plate, we have from equation (1)

$$s^2=s_0{}^2-\frac{4L^2E_0}{\nu}$$

at sufficiently small angles of deviation (up to 7°). s_0 is given by $\frac{Ln\lambda}{d}$, where L is the distance from the specimen to the plate, and λ is the electron wave-length in Å. It is evident that if E_0 is positive s is less than s_0, and so the spot is drawn inwards. If $\frac{4L^2E_0}{\nu}$ is greater than $s_0{}^2$ the reflexion cannot occur at all, on account of total internal reflexion. The change in the direction of the reflected beam depends upon surface being quite flat and smooth, and it is only in such cases that it can be appreciable. It is this requirement that makes the effect difficult to observe, since the surface must be flat to within order of magnitude of the actual interatomic distance. If the surface is in any way lumpy the electrons

will for the most part pass through the lumps and ridges, giving rise to a cross-grating pattern (single crystal) or rings (polycrystalline) similar to those obtained in transmission experiments. In such cases the deviation of the beam due to the refractive index is extremely small and could not possibly be detected using high-voltage electrons.

Professor Thomson * examined single crystals of Cu and Ag after etching and also after polishing, but found no trace of this effect. However, in the case of certain non-metallic substances having well-marked cleavage planes he found evidence of such a shift when reflecting high-speed electrons from the cleavage faces. For example, the various orders of the NaCl spectra were not uniformly spaced, but the innermost spectra were drawn inwards relative to the outer. Although there are a number of insulators which have good cleavage faces they are hardly suitable for such measurement of inner potential, because there is always the possibility of the surface becoming charged.

This is probably the reason why low-voltage electrons have given negative values, whereas high-speed electrons tend to give positive values. It is possible that the high-speed electrons give rise to ionization by sending off secondary electrons when they are impinging at small angles on the face of the crystal. The crystal is thus left positively charged and attracts to itself negatively charged ions, forming a double layer on the surface which would cause refraction effects in a manner similar to those we should expect from a definite internal potential.

The problem is investigated here by examining the diffraction effects from certain metallic cleavage faces using high-voltage electrons (about 30,000 volts). The specimen is freshly cleaved and mounted as quickly as possible on its support by means of Wood's metal to avoid contamination of the surface by exposure to air. A good electrical contact to earth is thus provided and difficulties due to charging up are avoided.

Nature of the Diffraction Effects.

The effects as observed experimentally are much more complex than a mere shifting of the position of the

* G. P. Thomson, Proc. Roy. Soc. A, cxxxiii. p. 1 (1931).

diffracted beam, since this would only be the case for a surface which is flat and smooth to a very high degree. If the surface is artificially polished we generally get diffuse rings, apparently due to a layer of amorphous material on the surface, and so a good cleavage face is the only surface that can be used when we are looking for this particular effect of refractive index.

As will be seen in this paper certain metallic cleavage faces do appear to be suitable, but even in these cases the effect is complicated, due to curvature and imperfect cleavage (lumpy surface). The diffraction image is determined by the superposition of three diffraction curves corresponding to three translations of the lattice.

If these curves have sharp maxima which are exactly superposed the resultant diffracted beam will occur in the direction given by the Bragg formula

$$n\lambda = 2d \sin \theta.$$

The width of these diffraction curves is determined not merely by the number of scattering centres along the respective translations, but also by a geometrical factor which depends on the inclination of the row of scattering points to the direction of the incident beam. Thus for an equal number of scattering centres the curves associated with those points arranged most nearly along the direction of the incident beam are much broader than those associated with the points arranged at a considerable angle to this direction.

If these individual diffraction curves are fairly broad the resultant intensity obtained by superposition will perhaps occur at a point not given by the Bragg condition. In the case of electrons the amplitude of the resultant wavelet scattered by a comparatively small number of atoms is so considerable that quite strong effects can be obtained from layers only a few atoms thick. In this respect electron diffraction differs very considerably from X-ray diffraction.

The effective number of scattering units lying in the plane of the crystal and normal to the direction of the incident beam is so great as a rule that the resolution in this direction is quite high, and the spots are constrained to lie on quite narrow lines normal to the shadow edge (this being the line in which the plane face of the specimen would intersect the photographic plate if produced).

The diffraction curves due to the translation along the direction most nearly parallel to the incident beam and that going into the depth of the crystal are sometimes quite broad, and so give rise to curious effects. The spectra are often considerably elongated in certain directions, and in some cases they will move over a fairly wide range as the angle of incidence is varied. In other cases they may be visible in a fixed position, but with varying intensity, for quite a large variation of glancing angle. This amounts to a considerable relaxation of the usual diffraction condition imposed by the Bragg equation.

The effects have been observed by a number of experimenters. G. P. Thomson * first drew attention to the wide range of angles over which reflexions are visible. Kirchner and Raether † published a number of photographs showing a remarkable elongation of the diffracted spots, and the effects have been recently discussed in more detail by Raether ‡.

It is important to note that the curve due to the lattice extension along the direction normal to the crystal surface is the only one to be displaced by the effect of refractive index. This means that we must have a reasonable depth of penetration into the crystal in order to determine E_0 with any degree of accuracy, otherwise the spectrum would be so extended that an accurate measurement of its position would be impossible. Also it is evident that a photograph taken at fixed incidence may give quite misleading results, since in such cases the maximum of intensity would appear at a point corresponding to the maximum of the resultant intensity curve given by superposing the two individual intensity curves.

For this reason rotation photographs are always taken before measuring the extent of the deviation. It can be shown that convex curvature has an effect very similar to that of rotation, and so is no disadvantage provided the curvature does not involve a disruption of the surface layers, as it might if it was too great.

If the surface is lumpy the electrons would pass almost entirely through the lumps, and they would give spots at the undeviated positions, but if the surface

* G. P. Thomson, Proc. Roy. Soc. A, cxxxiii. p. 19 (1931).
† F. Kirchner and M. Raether, *Phys. Zeits.* xiii. p. 510 (1931).
‡ H. Raether, *Zeit für Physik*, lxxviii. p. 527 (1932).

is only partially lumpy with considerable plane areas here and there (see fig. 7) we might expect to observe the intermediate cases, and thus obtain a spot drawn inwards from the undeviated to the full deviated position. Such an effect would in general be superposed on the elongation due to low resolving power, and so would not be a good indication of the extent of the shift due to the inner potential. However, if the elongation occurs inwards from the undeviated position it seems very likely that it is actually due to the effect of refraction, because the effect due to low resolving power should be distributed more or less equally on either side of the intensity maximum.

Apparatus.

The electron diffraction apparatus used for the reflexion of an electron beam from the metallic cleavage faces was similar to one of those originally designed by Professor

Fig. 7.

Rough surface with flat peaks.

G. P. Thomson *. No additional features were involved here beyond the use of a hot cathode in place of the gas discharge tube.

Preparation of Single Crystals.

The single crystals of zinc, antimony, bismuth, and tellurium were prepared according to the method described by Bridgman †, lowering a tube containing the molten metal slowly through the bottom of a vertical tubular furnace. The purity was in all cases not less than 99·85 per cent. The speed of lowering was 1 cm. an hour for Zn and Bi, 10 cm. an hour for Sb, and 25 cm. an hour for Te. Pyrex glass was used for Bi, Te, Zn, and combustion tubing for Sb on account of its higher melting-point. It was found to be advisable to grease the combustion tubing as described by Bridgman, but this did not seem to be necessary in the case of the pyrex

* G. P. Thomson and C. G. Fraser, Proc. Roy. Soc. A, cxxviii. p. 641 (1930).

† P. W. Bridgman, Proc. Amer. Acad. Science, lx. p. 305 (1925).

tubing. The lowering was effected by means of a water-float, and the speed could be adjusted within fine limits by regulating the flow of water into the tank.

The furnace was run in general about 75° C. above the melting-point of the metal. Large castings about 8 mm. diameter could be obtained fairly readily from Zn and Sb, but only small castings of about 5 mm. diameter were consistently successful in the case of Bi and Te.

The thickness of the walls of the tubing appears to be a very important point and should be as thin as possible consistent with the strains involved.

Experimental Results.

Antimony.—A large number of cleavage slips of Sb were examined, and in a great many cases nothing could be observed on the plate. It is remarkable that it should be so much more difficult to obtain diffraction effects from metallic surfaces than from such substances as rock-salt, mica, etc.

About thirty per cent. gave definite effects, but of these again only a small number gave what appears to be definite evidence of refractive index.

Photographs were taken at fixed incidence corresponding to maximum intensity of the observed effects, and then at the smallest possible glancing angle to look for the first-order spectra. This is of importance, because the inner potential will in general be sufficient to prevent the lower-order spectra from being observed, as already explained. Finally a rotation photograph was taken through approximately 2°. Fig. 1 (Pl. XIX.) illustrates a drawing out of the spot, and was taken at fixed incidence. It will be seen that the third order is drawn out so as to merge into the second order, and the intensity is considerably greater at the head of the spot corresponding to the undeviated position.

The effect on the screen was sensitive to alteration of incidence, suggesting that curvature was playing little or no part here. In addition, curvature could not be detected by applying simple optical tests of reflexion.

It is not possible to determine E_0 from such a plate, owing to the running together of the second and third orders.

Rotation photographs gave several higher orders,

and then definite limits could be ascribed to the extension, thus allowing a value for E_0 to be determined.

Two such photographs are reproduced in figs. 2 and 3 (Pl. XIX.).

To deduce a value for μ we have, substituting in equation (1),

$$n^2 = \frac{d^2 s^2}{L^2 \lambda^2} + \frac{(\mu^2 - 1)4d^2}{\lambda^2},$$

where L is the distance from the specimen to the photographic plate and s is the observed distance of the diffracted spot from the central spot. If n^2 is plotted against s^2, the gradient gives $\frac{d^2}{L^2\lambda^2}$, from which d can be found if λ is known, and the intercept gives $\frac{(\mu^2-1)4d^2}{\lambda^2}$, from which μ can be determined. The method is useful where d is unknown, and sharp displaced reflexions are observed, but in the cases investigated here it proved a little ambiguous, and we proceed as follows :—

From equation (1) we have

$$\frac{\lambda}{2 \sin \theta} = \frac{d}{n}\left(1 - \frac{8E_0 med^2}{n^2 h^2}\right)^{-\frac{1}{2}}.$$

Hence $\frac{\lambda}{2 \sin \theta}$ is independent of λ even when E_0 is finite and not zero. We plot $\frac{\lambda}{2 \sin \theta}$ against E_0 for values of E_0 from 0–18 volts for the different metals concerned. The curves for Sb are reproduced in fig. 8 for the first up to the seventh order.

The observed values of $\frac{\lambda}{2 \sin \theta}$ are then marked off along a strip of paper (if the spot is extended the full limits of its extension are marked off). This strip is then moved over the curves until a suitable fit is observed, and the corresponding value of E_0 read off directly. In some cases a striking fit of the observed position was found.

The fourth-, fifth-, and seventh-order reflexion from the (111) planes of antimony gave a good fit for $E_0 = 12$ volts,

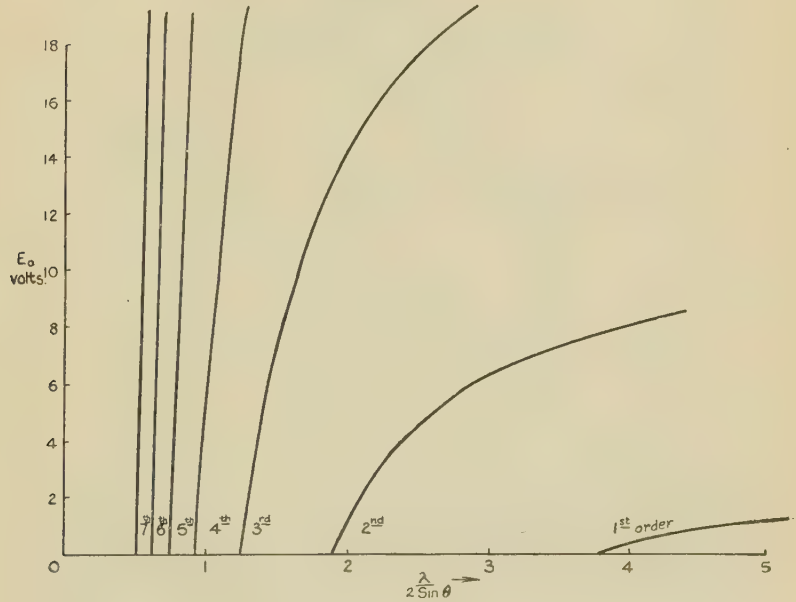
and the individual values, taking the mean for two separate plates, were :—

TABLE I.

Order.	4th.	5th.	7th.	
E_0	12·5	11·4	12·8	Mean 12·2 volts.

It is improbable that this should be in error by more than ± 1 volt. The observed values of $\frac{\lambda}{2 \sin \theta}$ for the undeviated head of the spots are given in Table II.

Fig. 8.



E_0 plotted against $\frac{\lambda}{2 \sin \theta}$ for antimony.

The sixth order is very feeble; the reason for this is unknown. It will be seen from fig. 8 that the third order nearly merges in the second for $E_0 = 12$ volts, and this gives a vivid picture of the actual magnitude of the effect for a comparatively small inner potential. The extension measured above is believed to be due to refractive index for two reasons: (1) the extension occurs from the undeviated position inwards, and not on either side

of this position; (2) the amount of the extension decreases for the higher orders in accordance with what we should expect for such an effect (this is seen by the appreciably constant values for E_0 in Table I.). This value of 12·0 volts for E_0 was confirmed also by a series of four photographs taken from a very clean and smooth cleavage surface of Sb, for which the fifth-order reflexion always appeared at a position corresponding to the full deviation to be expected due to an inner potential of 11·4 volts. The previous values are probably a little more reliable, and so the experiments may be considered

TABLE II.

Reflexion from 111 Planes of Sb.

Rotation Photographs.

Order.	$\frac{\lambda}{2 \sin \theta}$. Fig. 3 *a*. λ=·066 Å.	$\frac{\lambda}{2 \sin \theta}$. Fig. 2 *a*. λ=·070 Å.	$\frac{d}{n}$ calculated for d=3·76 Å.
1....	—	—	3·76
2....	1·88	1·85	1·88
3....	1·24	1·27	1·25
4....	·95	·96	·94
5....	·76	·76	·75
6....	—	—	·63
7....	·55	·54	·54

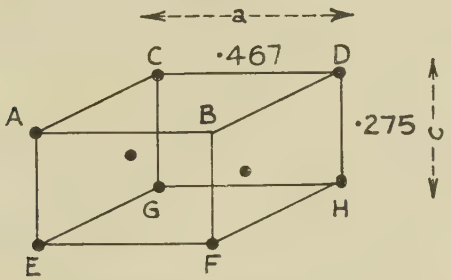
as giving E_0=12 volts for antimony. In figs. 2 *a* and 3 *a* a cross-grating pattern can be seen in addition to the reflexion from the cleavage plane. This is geometrically the same as would be produced by transmission through the lumps on a rough surface. There are, however, interesting physical differences which are due to the different effects of the resolving power in the two cases.

Professor Thomson * has shown that these cross-grating effects can be readily interpreted by use of the reciprocal lattice. The pattern observed by diffraction is geometrically equivalent to that obtained by taking a section of the reciprocal lattice by a plane normal to the direction of the incident beam. The reciprocal lattice of antimony is the hexagonal cell (fig. 9) with

* G. P. Thomson, Proc. Roy. Soc. A, cxxxiii. p. 1 (1931).

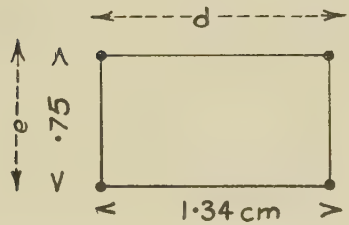
$x=\cdot275$, $a=\cdot467$. If the beam travels normal to the plane CDGH we get the symmetrical pattern (fig. 2*a*), and

Fig. 9.



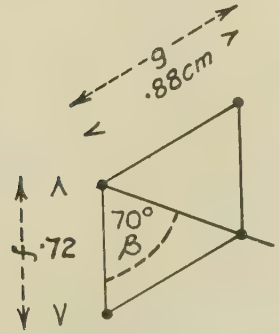
Hexagonal reciprocal lattice for Sb.

Fig. 2*a*.



Unit of grating pattern (symmetrical).

Fig. 3*a*.



Unit of grating pattern (asymmetrical).

if normal to ADEH we get the asymetrical pattern (fig. 3*a*). The observed and calculated values of the translations are given in Table III., and drawings of the pattern

unit are given in figs. 2*a* and 3*a*, corresponding to photographs 2 and 3 (Pl. XIX.).

Zinc.—The cleavage surfaces of Zn were remarkably smooth but in most cases slightly curved. The photographs of which fig. 4 (Pl. XIX.) is typical shows spots,

Table III.

Fig. 2*a*. Symmetrical.		Fig. 3*a*. Asymmetrical.	
Obs.	Calc.	Obs.	Calc.
$d=1{\cdot}34$	1·31	$f={\cdot}72$	·73
$e={\cdot}75$	·77	$g={\cdot}88$	·86
$\alpha=90°$	90°	$\beta=70°$	71° 10′

rather elongated, but the intensity is a maximum at the mean position and fades away to either end, differing in this respect very much from Sb. In Table IV. the values of $\frac{\lambda}{2\sin\theta}$ are given for the intense part of the spot in column 2 and for the extreme limits in column 3. The values of $\frac{d}{n}$ for the 0001 planes of Zn are given in column 5. The results are the mean of several plates for which the wave-length ranged from $\lambda={\cdot}065$ to ·075 Å.

Table IV.

Spot.	$\frac{\lambda}{2\sin\theta}$. Intense spot.	$\frac{\lambda}{2\sin\theta}$. Limits.	N order.	$\frac{\lambda}{2\sin\theta}$. $E_0=0$.	$\frac{\lambda}{2\sin\theta}$. $E_0=15{\cdot}5$ volts.
	—	—	2	2·47	Absent.
1	2·07	2·27–1·87	4	1·23	2·04
2	1·02	1·06– ·99	6	·82	·99
3	·70	·72– ·68	8	·62	·67

The surface was then examined by X-rays, and the usual spacing of the cleavage planes 2·47 Å was confirmed. This additional check was considered essential here because the observed values of $\frac{\lambda}{2\sin\theta}$ are so displaced

inwards that they are in the ratio 1 : 2 : 3, although they are actually fourth-, sixth-, and eighth-order spectra respectively. The odd order reflexions from the 0001 planes are, of course, absent. The surface was afterwards etched, and the spectra appeared then in their normal undeviated positions corresponding to a spacing of 2·47 Å.

Thus in the case of zinc there is quite definite evidence of a full shift of the reflexions due to the effect of inner potential, and although the spots here are also slightly extended, due to low finite resolving power, nevertheless the extension leads to no ambiguity in this case, since they are extended on either side of their deviated position, and the intensity has fallen to zero where the undeviated spot would lie. The best value deduced graphically is $E_0=15{\cdot}5$ volts, and the theoretical values of $\frac{\lambda}{2\sin\theta}$ using this figure are given in column 6 for comparison with the observed values and the undeviated values for $E_0=0$ (column 5). The value 15·5 volts is in remarkably good agreement with the value obtained by Lass and Rupp * (15·3 volts) using low voltage electrons.

Bismuth.—A large number of cleavage slips of bismuth were examined, and in most cases only the second-order reflexion from the cleavage plane (111) could be observed. Some photographs gave rather extensive cross-grating patterns (Pl. XIX. fig. 5), but the spots were sharper than in the case of Sb and did not change much in extent or intensity when a rotation photograph was taken. In the case of three photographs only was there a definite evidence of extension of the spot towards the shadow edge as observed for Sb.

Rotation photographs from these specimens, however, failed to bring out the higher orders, and so it is impossible to deduce any definite value for E_0 as in the case of Sb.

It seems probable, in fact, that in this case the extension is due merely to low resolving power as a consequence of small depth of penetration.

The appearance of the cross-grating pattern suggests that the bismuth cleavage surface is rather lumpy, and examination under the microscope revealed deep but narrow furrows running across the surface. These appear very often on a freshly cleaved Bi surface and are

* J. Lass and E. Rupp, *Annalen der Physik*, v. p. 611 (1931).

probably associated with the secondary structure *. The cross-grating patterns could be explained precisely as in the case of Sb with $c = \cdot 261$, $a = \cdot 442$. The asymmetric patterns alone were photographed, the effects being even more feeble than in the case of Sb.

Tellurium.—Excellent cleavage faces of Te were obtained and examined in the usual way. A typical rotation photograph is shown in fig. 6 (Pl. XIX.). Although the spots on the equator line are drawn inwards a little towards the shadow edge, it is again very doubtful whether this can be due to any refractive effect. No photographs were obtained giving evidence of a complete shift of the different orders as in the case of Sb and Zn. The observed $\frac{d}{n}$ values are given in the following table, along with the values for the cleavage plane, which is here 1010.

Table V.

Spot.	Obs. $\frac{d}{n}$.	Calc. $\frac{d}{n}$.	Spacing $d = 3 \cdot 84$ Å.
1	1·93	1·92	
2	1·26	1·28	
3	·94	·96	

Tellurium has a structure very similar to Sb and Bi, but differs very much in so far as the plane of easy cleavage is parallel to the rhombohedral axis and not perpendicular to it as in the case of Sb and Bi (111 plane in both cases). When freshly cleaved, it gives most brilliant effects on the fluorescent screen, but has a most remarkable property of soon losing such power. The change occurs in air and just as quickly *in vacuo*, and even if the electron beam has never impinged upon its surface. It appears that the surface undergoes some kind of spontaneous disintegration, as subsequently only very diffuse and ill-defined rings were obtained, which so far cannot be identified. This unfortunately restricts the work that can be done with tellurium, as a freshly cleaved surface must be used each time. Eight surfaces were

* F. Zwicky, Phys. Rev. xl. p. 63 (1932).

examined, but in all cases the spots were in the undeviated Bragg position.

Summary.

Definite evidence of refraction due to an inner potential has been found when high speed electrons are reflected from cleavage faces of zinc and antimony, but not from bismuth and tellurium. The experiment demands a very exacting perfection of the cleavage faces, which is difficult to attain in the case of metal single crystals. The values of the inner potential deduced from the present work are antimony $E_0 = 12$ volts and zinc $E_0 = 15{\cdot}5$ volts. These are believed to be accurate to within at least ± 1 volt in each case.

In conclusion, I must thank Professor G. P. Thomson for suggesting the problem and for useful advice throughout the course of the work; also Dr. Müller, of the Royal Institution, who kindly took the X-ray rotation photographs from the cleavage planes. The author must also thank the Department of Scientific and Industrial Research for an award during the tenure of which this work has been carried out.

The Imperial College of Science and Technology, London.
August 1933.

LXVII. *Note on the Gold-leaf Electroscope.* *By* J. Back, *The Washington Singer Laboratories, University College, Exeter* *.

VARIOUS types of electroscopes for the measurement of small electrical charges have been designed, but, on the whole, they may be divided into two classes, viz., those employing a gold-leaf and those in which the moving part is a quartz-fibre. Thus Kaye † used a modified form of Wilson's tilted leaf electroscope, and obtained a sensitivity of 5·5 mm. per volt over the range 0·8–1·20 volts with a plate potential of 207 volts. With a double-plate electrometer, in which a quartz-fibre 0·003 mm. diameter was anchored at either end, Laby ‡

* Communicated by Professor F. H. Newman, D.Sc.
† Proc. Phys. Soc. xxiii. p. 209 (1911).
‡ Proc. Cambridge Phil. Soc. xv. p. 106 (1909).

found that for a potential difference of 60 volts between the plates, the latter being 1 cm. apart, 1 volt applied to the fibre gave a deflexion of 0·2 mm. Lindemann and Keeley * devised a torsional quartz-fibre electrometer, in which two silvered needles, capable of rotation about a horizontal axis, are mounted perpendicularly to the torsion fibre. These needles move in an electric field between two pairs of fixed plates, and their movement is constrained by the torsion of the suspended fibre. With a fibre 6 μ diameter and 1·4 cm. long a sensitivity of 0·76 mm. per volt was obtained with a plate potential difference of 97 volts, the plate separation being 0·6 cm. Barber † designed a double plate electrometer, employing a single suspended quartz-fibre as the moving system. With 60 volts on the plate the sensitivity was 2·60 mm. per volt.

There are three important points to be considered when designing any type of electroscope. In the first place, the moving system should be of such a nature that it can be replaced, easily and readily, after fracture. It is a troublesome matter to renew a gold-leaf, although after practice and much waste of leaf it can be done. With the quartz-fibre type of instrument the fibre must be platinized, or made conducting in some way, and, in addition, there must be good electrical contact between the fibre and its support. This means that the sputtering process has to be carried out with the fibre attached to its support. Secondly, the fibre must be very sensitive ; and, thirdly, it is desirable that the relationship between the sensitivity and the fibre potential shall be a linear one.

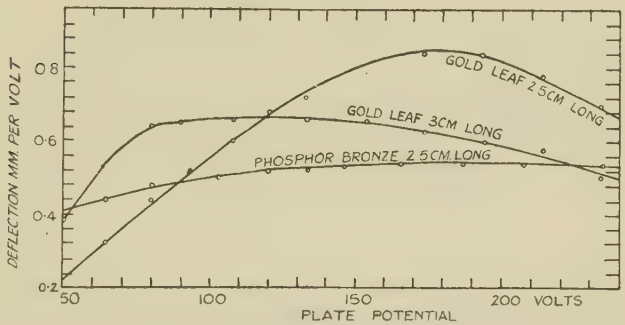
At present it is possible to obtain phosphor-bronze strip of extremely small cross-sectional area, and Professor Newman suggested that it might be used instead of gold-leaf in the Wilson type of tilted electroscope. The strip employed in the present work was 0·005 cm. wide, breaking stress 55 grams, and electrical resistance 3·2 ohms per cm. The strip was first tried in the form of a straight length, but this was found to be insensitive. A better type of strip was one in the form of a small spiral of about six turns, the axis of the spiral being perpendicular to the direction of the strip's displacement. The spiral extended over a length of about 1 mm. from the point at which

* Phil. Mag. xlvii. p. 577 (1924).
† Phil. Mag. vi. p. 458 (1928).

it was soldered to the brass supporting-rod, the remainder of the strip being straight. This arrangement decreased the restoring torque on the fibre, and the arrangement was most sensitive when the straight portion was parallel to the plate at which the fixed potential was applied.

The instrument was calibrated directly by applying known potentials to the strip, the potential on the plate being constant. Measurements of the resultant deflexions were made by observations of the image in the field of a telemicroscope, twenty-five divisions of the eye-piece scale corresponding to an actual displacement of 1·0 mm. Readings were taken for a strip potential of 2 volts when the plate potential was varied over the range

Fig. 1.



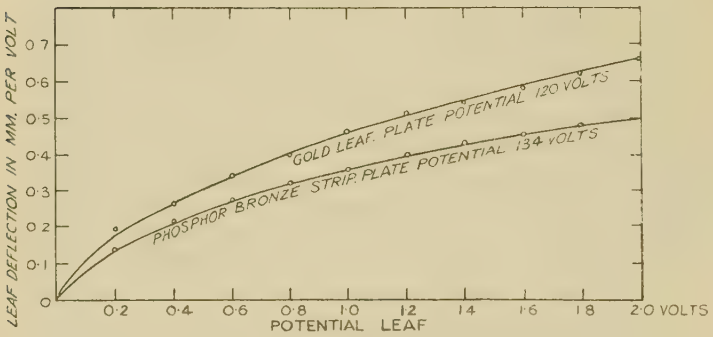
Sensitivity of electroscope (potential of leaf constant).

50–250 volts. Similar observations were made with a gold-leaf, and the two sets of sensitivity curves are shown in fig. 1. It will be noted that the sensitivity of the phosphor-bronze strip, although less than that of the gold-leaf, is approximately constant over the range 50–250 volts, and, in addition, is approximately 60 per cent. of that of the gold-leaf at the latter's most sensitive plate voltage.

In a second set of experiments the deflexions of the strip for a particular plate voltage, corresponding to increasing strip potentials, were observed and compared with those for a gold-leaf. Fig. 2 shows the results. The relationship between the sensitivity and strip potential was found to be a linear one for the higher strip potentials, and the sensitivity was not greatly inferior to that of the gold-leaf.

The phosphor-bronze strip is thus, on the whole, somewhat less sensitive than the gold-leaf, but it is much easier

Fig. 2.



Sensitivity of electroscope (plate voltage constant).

to mount, has a more uniform sensitivity, and is more easily detached without fracture from the plate if it should " fly " to the latter.

LXVIII. *Electron Diffraction Patterns from Platinized Asbestos.* (*Abridged Thesis approved for the Degree of M.Sc. by the University of London.*) *By* D. A. Richards, *M.Sc., A.R.C.S., D.I.C.* *

[Plates XX.–XXII.]

Abstract.

Electron diffraction patterns have been obtained from platinized asbestos, the platinization being effected by reduction of platinic chloride in hydrogen. Increasing quantities of platinum were added and the transition from the asbestos diffraction pattern to that corresponding to platinum observed. A quantitative estimation of the ratio of platinum to asbestos in the final state of the specimen was made, and also an approximate determination of the upper and lower limits of the thickness of

* Communicated by Prof. G. P. Thomson, F.R.S.

the platinum deposit, and thence an estimation of the area of asbestos covered by the deposit. The large surface of asbestos required was consistent with the diffuse rings obtained in the diffraction pattern after the platinum had been dissolved in aqua regia. Possible explanations why the platinum structure does not appear until a high degree of platinization has been effected are considered, and the observed phenomena appear to indicate that the platinization results in a splitting of the fibres, with the result that a fresh surface of asbestos is presented to the electron beam, nullifying the effect of the additional platinum.

Apparatus.

THE electron camera was of the usual cold cathode type described in G. P. Thomson's earlier papers * and designed for transmission experiments, the only difference being that a small brass frame was provided across which the fibres were stretched. This framework could be moved in or out of the beam by means of a ground-joint, so that the necessary adjustment could be made from the outside without altering the vacuum conditions in the apparatus.

Since the voltage across the discharge-tube is a function of the pressure in the tube, it is essential that the latter should remain constant and controllable if sharp diffraction pictures are to be obtained. This condition was realized by using a needle-valve surrounded by air at low pressure. It was found that a convenient pressure was that of the oil-pump which backed the Gaede rotary pump and mercury-vapour pump of the evacuating system.

The high voltage was applied across the discharge-tube by means of a large induction coil, the current being rectified by a Phillips's Mettalix valve. The applied voltage was steadied by means of a high-water resistance in series with the high-tension circuit, and a high-voltage condenser of 0·002 mf. capacity shunted across the discharge-tube.

The voltage was determined by measuring the sparking distance between two aluminium spheres 5 cm. diameter shunted across the discharge-tube.

* G. P. Thomson, Proc. Roy. Soc. A, cxvii. p. 600 (1928).

Electron Diffraction Pattern from Asbestos and Platinized Asbestos.

That platinized asbestos gives a diffraction pattern corresponding to the asbestos structure alone was observed by G. P. Thomson *. This result is obtained even when the platinum is present in sufficient quantity to render the specimen almost black.

It seemed desirable first to correlate the diffraction pattern obtained from asbestos by electron diffraction with that obtained by the X-ray investigation which had been published shortly after the research herein described had been commenced †.

Specimens of unplatinized chrysotile, in the form of parallel fibres, were mounted on the brass framework so as to be at right angles to the electron beam. Since the unit cells of chrysotile are arranged in the fibre, with the short "*c*" axes in the direction of the fibre, and the other two axes oriented at random in a plane approximately at right angles to the direction of the fibre, diffraction photographs of the well-known rotation type should be obtained. In practice, however, many of the photographs obtained showed a transition between the "rotation pattern" and the "Debye Scherrer ring pattern," the explanation being that, owing to the small penetration of the electron waves, only the surface conditions of the specimen are investigated, and hence any slight disturbance of the surface, such as would be produced by splitting and preparing the specimen, would tend to give the latter pattern instead of the former (fig. 1, Pl. XX.). However, a specimen was obtained which gave a "rotation pattern" free from rings (fig. 2, Pl. XX,), and, by bringing a magnet near the beam so as to deflect the central spot to the edge of the plate, layer lines of a high order were obtained (fig. 3, Pl. XX.). In the photograph the eighth layer line is visible, while the tenth layer line was visible on the original plate.

A reciprocal lattice diagram ‡ was drawn, using the data given by Bragg for the lengths and angles of the unit cell, and on this was superposed an enlarged scale drawing of

* G. P. Thomson, Proc. Roy. Soc. A, cxviii. p. 625.
† B. E. Warren and W. L. Bragg, *Zeit. f. Krist.* lxxvi. p. 201.
‡ *Vide* J. Bernal, Proc. Roy. Soc. A, cxiii. p. 117.

the "rotation" photographs obtained by electron diffraction, the scale being so arranged that the layer lines coincided. It was then possible to assign indices to the planes giving strong reflexions on the plate, and these were found to be in close harmony with those mentioned by Bragg.

The absolute spacing of the (063) ring was determined for twelve plates from the spark-gap measurements, and agreed with Bragg's value to slightly less than 1 per cent., this error being due to the uncertainty with which the voltage can be determined. This measurement was eliminated by taking photographs of the asbestos pattern and that of gold on different parts of the same plate. Comparison of the diameter of the (063) ring of asbestos and the (220) ring of the gold pattern enabled the (063) spacing to be determined, as the size of the unit cell of gold is known to a sufficient degree of accuracy. By mounting the two specimens in adjacent positions on the cone, the two photographs could be taken in rapid succession, *i. e.*, within two minutes, so that the chance of voltage variation was small. The agreement by this method was 0·4 per cent.

It appears, therefore, that electron diffraction of chrysotile asbestos agrees with the X-ray investigation both as regards the spacings of the planes giving strong reflexions and the indices of these planes.

Electron Diffraction from Platinized Asbestos.

Considering once more the diffraction patterns obtained with highly platinized asbestos, four possible explanations present themselves:—

(*a*) That suggested by Thomson*, namely, that the platinum is deposited in cracks in the asbestos where the platinum would not reach it.

(*b*) That the platinum is amorphous, and would not give sharply-defined rings of the well-known platinum structure.

(*c*) That the deposit is crystalline and on the surface, but of relatively large lumps of about 10^{-4} cm. thick, and consequently opaque to the electron beam.

* Proc. Roy. Soc. A, cxviii. p. 653.

(*d*) That the platinum becomes in some way associated with the asbestos structure without appreciably altering the size of the unit cell.

It was suggested that a possible solution to the problem might be obtained by continued replatinization of the same specimen and its examination in the electron camera at regular intervals. If the first explanation were correct then ultimately the specimen would become so saturated with platinum that all the cracks would be filled up and a platinum diffraction would result. If the second explanation were true, then the amorphous deposit would so cover the asbestos as to mask completely its surface, and no diffraction would result. If the third explanation were true, then no diffraction pattern would result, as the lumps of platinum would be too large, but a great deal of "background" scattering, just as a thick film in a transmission experiment gives a considerable amount of inelastic "background" scattering. Even if there were a few portions of the specimen of dimensions suitable to give an electron diffraction pattern, the diffraction rings resulting therefrom would be lost against the strong background. If the fourth explanation were true, then some change in the relative intensities of the rings would be expected owing to the larger scattering power of the platinum in comparison with that of the lighter atoms of the ordinary asbestos structure. Of the possible explanations the last is the most unlikely.

Platinization was effected by the reduction of platinum tetrachloride in a stream of hydrogen at about 100° C. The specimen was saturated with 0·5 per cent. platinic chloride, dried, and then heated in a stream of hydrogen in a water-bath. Uniform distribution of the platinum was aimed at rather than addition of known amounts of platinum, as a knowledge of the amount of platinum present after each platinization has little practical value, as it is the surface distribution, and not the volume distribution, which is of importance, and the surface distribution, demanding as it does a knowledge of the total area of asbestos exposed, itself probably a varying quantity, cannot be determined.

It was found essential to sweep the air out of the apparatus with a stream of carbon dioxide as the catalytic activity of the platinized asbestos was so great, that

when the hydrogen was passed over the specimen it combined with the atmospheric oxygen, the heat of reaction being sufficient to render the specimen red-hot and ignite the mixture with explosive violence. At red-heat the asbestos structure is destroyed.

After twenty successive platinizations only the asbestos pattern showed (fig. 4, Pl. XX.), although the specimen was jet-black and had lost a great deal of its original woolly nature, probably due to the repeated splitting of the asbestos when it is soaked in the solution.

This loss of the woolly nature seemed to suggest that each process of platinization resulted in a splitting of the asbestos so that a fresh surface was exposed, and so an asbestos pattern was obtained. In addition, it was observed that after the specimen was boiled in distilled water a cloud of finer fibres was suspended over the main bulk of the specimen, which sank to the bottom. It seemed reasonable to suppose that, if these fibres could be isolated and platinized, they should show the platinum structure (if the deposit were crystalline), as they would probably be incapable of further splitting. A small quantity of this cloud was sucked up into a narrow tube and filtered through a cobweb stretched over the end of a piece of glass tubing. The fibres were retained on the web and the water drained away. The combined cobweb-platinized asbestos specimen was mounted in the apparatus and showed only the asbestos structure, despite the fact that the fibres (together with the main bulk of the apparatus) had been platinized successively twenty times. The composite specimen was then replatinized twice—making a total of twenty-two times in all—and showed the platinum structure (fig. 5, Pl. XXI.). This showed clearly that the platinum deposit was crystalline, and appeared to indicate that the reason why the platinum failed to show itself previously was due to the successive splitting of the fibres. The latter, however, was not conclusive evidence, as the platinum deposit might be on the cobweb and not on the asbestos. It remained, therefore, to continue the replatinization with the main bulk of the specimen. In the photographs taken after the 20th, 22nd, 24th, and 25th platinizations only the asbestos rings appeared. After the 27th platinization faint extra rings, which on measurement were found to correspond to the platinum structure, were visible. After

twenty eight platinizations these rings increased in intensity so as to be about equal to the (063) ring of asbestos (fig. 6, Pl. XXI.), while after twenty-nine platinizations only the platinum rings appeared in the diffraction pattern, with the exception of the (063) and the (061) and 202 group of rings, due to asbestos, which were very faintly visible (fig. 7, Pl. XXI.).

Determination of the Amount of Platinum present in the Final State of the Specimen.

A gravimetric estimation of the amount of platinum present in the final state of the specimen was made by weighing out a small quantity of the platinized asbestos, dissolving away the platinum in aqua regia, and weighing the residue of the asbestos. It had been ascertained previously that the asbestos was unaffected by aqua regia, as a small quantity of asbestos wool, after boiling with the acid, gave the diffraction rings corresponding to asbestos, and no residue was obtained when the acid was evaporated to dryness on a watch-glass.

The ratio of asbestos to platinum which is necessary before the platinum shows was found to be 1 : 1·8 by weight or 1 : 0·26 by volume.

Examination of the Residue of Asbestos after the Platinum has been removed.

A small quantity of the platinized asbestos from which the platinum had been dissolved was inserted in the apparatus and a diffraction pattern obtained. Instead of the expected asbestos pattern, only a diffuse ring was obtained (fig. 8, Pl. XXI.). In one case, however, a faint outer ring was observed which, on measurement, proved to be the (063) ring of asbestos. This showed conclusively that the diffuse ring was due to asbestos, a fact which might have remained doubtful if only the small diffuse ring were available for measurement, owing to the difficulty of measuring the diameter of the ring accurately.

Considering once more the possible explanations for the absence of the platinum diffraction pattern, except after considerable platinization, the fourth explanation—that the platinum becomes associated in some way with the asbestos—may be eliminated as unlikely, as no change in the relative intensities of the rings was observed even

after twenty-four platinizations. The second possible explanation—that the deposit is amorphous—is eliminated, as the platinum diffraction pattern was ultimately obtained. The third explanation—that the deposit is crystalline, but of relatively large lumps of the order of 10^{-4} cm., and consequently opaque to the electron beam—may be considered in the light of the photomicrographs (figs. 9 and 10, Pl. XXII.) of the highly platinized asbestos, and an identical specimen from which the platinum had been dissolved. These were kindly taken by Mr. B. K. Johnson, A.R.C.S., B.Sc., D.I.C., from slides prepared by the Author. Since the magnification is 1030, one would expect lumps of platinum of the order of 10^{-4} cm. to show as lumps of 1 mm. edge on the photograph. The edges of the fibres appear fairly smooth, however, and any slight irregularities are less than 1 mm. in magnitude. Some of the fibres are only 1 mm. in diameter on the plate, so that the size of the deposit on the specimen must be less than 10^{-4} cm.

The first explanation, therefore, appears to satisfy the observed phenomena, namely, that the platinum is deposited in cracks in the asbestos and additional platinization causes a splitting of the asbestos so that a fresh surface is presented to the electron beam which nullifies the effect of the additional platinum deposited. That the splitting of the asbestos is not due entirely to the wetting and drying of the specimen is shown by the fact that, although some asbestos wool was boiled in distilled water and dried successively as many as thirty-six times, it showed the ordinary asbestos diffraction pattern, and not the diffuse rings which were obtained after the platinum had been dissolved from the platinized specimen.

When the specimen is immersed in the platinic chloride solution, presumably some of the solution is drawn into the interstices of the asbestos by capillary action. After drying, the platinic chloride is reduced in hydrogen with the formation of hydrogen chloride—two volumes for every volume of hydrogen—and thus the platinic chloride might be the seat of forces sufficiently great to cause splitting. If the cohesion between the platinum-asbestos surface is greater than that between adjacent asbestos fibres (whose small binding forces in the directions of the "*a*" and "*b*" axes are well known), the splitting which occurs to relieve the forces previously indicated is

more likely to occur between two asbestos surfaces than between a platinum-asbestos surface. This condition would be realized if the splitting occurred at right angles to the direction of the crack into which the platinic chloride is drawn. A small amount of platinum may be presented to the beam, but its diffraction pattern will be lost in the background scattering of the asbestos. Only when the amount of surface of platinum and asbestos are comparable—as occurred after twenty-eight successive platinizations—will the two diffraction patterns appear on the plate, and this appears to occur only after the asbestos has been split up into fibres sufficiently small to give a diffuse ring diffraction pattern. Any further platinization results in a preponderance of platinum on the surface of the specimen, with the result that only the platinum diffraction pattern is obtained.

Calculation of the Size of the Grains of Platinum Deposit.

In the case in which the diffraction pattern due to the platinum structure was observed, resolution was obtained between the (111) and (200) rings, while the (311) and (222) rings were unresolved. It is possible to calculate the upper and lower limits of unit cells in the little blocks of platinum deposited on the asbestos on the following assumptions:—

(1) That the limit of resolution is reached when the half-maximum value of the intensity of one ring coincides with that of the next.

(2) That the electron beam is monochromatic, *i. e.*, that the voltage across the discharge-tube is constant. The sharpness of the asbestos diffraction photographs taken at the same time as the platinum photographs suggest that the voltage fluctuations are small. In any case the assumption affects only the upper limit of the size of the platinum grains.

In order that the (111) and (200) rings may be resolved, it was calculated that the size of the grains must be more than 15 Å. or 4 unit-cells cube, and it must be less than 30 Å. or 8 unit-cells cube, if the (311) and (222) rings cannot be resolved with a perfectly monochromatic beam.

Assuming a value of 6 unit cells each way for the grains of platinum, it is possible to calculate the area of asbestos covered by one grain of platinum. Since the volume of asbestos associated, in the final state, with a given volume of platinum was determined, the mean area of cross-section of the asbestos required to give the calculated area was calculated upon the assumption that the splitting of the fibres occurred in the direction of the "*a*" and "*b*" axes only. The number of unit cells of asbestos, perpendicular to the fibre-length, obtained by this calculation was twenty-one. This figure seems high when one considers the diffuseness of the asbestos rings after the platinum had been dissolved away from the surface. In view, however, of the roughness of some of the assumptions made in the calculations, the figure is fairly satisfactory, and of the right order. While no great reliance can be placed upon the actual figures deduced, it is claimed that they are, at least, consistent with the photographs.

In conclusion, the author wishes to express his gratitude to the London County Council and to the Department of Scientific and Industrial Research for financial assistance during the period in which the research was conducted; to the Authorities of the Royal College of Science for the research facilities provided; to Mr. B. K. Johnson for the photomicrographs; and in particular to Professor G. P. Thomson, F.R.S., under whose kindly supervision the research was conducted, not only for his technical assistance, but for his inspiration and encouragement during those periods when the research appeared fruitless.

LXIX. *The Acoustical Theory of Exhaust Silencers of the Perforated Baffle Type.* *By* A. H. Davis, *D.Sc., Physics Department, National Physical Laboratory* *.

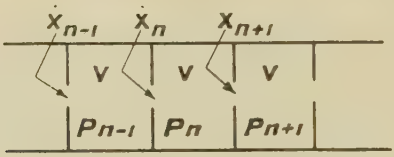
G. W. STEWART has shown that when a series of resonators or of apertures are arranged as side-branches to an acoustical conduit the arrangements constitute acoustical filters, the former tending to suppress all frequencies above a certain pitch and the latter to remove all below a certain critical frequency. So far as the writer is aware, however, no formal theory has been

* Communicated by the Author.

given of the acoustical action of the baffle-type silencers which are employed for silencing the exhausts of engines. Such silencers consist (fig. 1) of a series of perforated baffle-plates inserted in a conduit, and the following considerations show that they have the acoustical characteristics of low-pass filters.

The periodic discharge of exhaust gases into the silencer can be resolved into (*a*) a steady flow of gas, upon which is superimposed (*b*) an oscillatory flow of complex wave-form. In what follows it is taken for granted that the velocity of the steady flow of gas is small compared with the velocity of sound, so that the progress of sound-waves through the silencer is not greatly influenced by the steady flow. The oscillatory movements, however, may be resolved into their Fourier components, and the aim of the acoustical theory of the

Fig. 1.



Perforated baffle silencer.

silencer is to determine which of these components, if any, will be attenuated in progress through it.

For simplicity the silencer is considered to be infinitely long and to have an infinite number of sections, and in the first instance each baffle is assumed to contain one hole only. The radius of the hole is denoted by "*a*," the volume of the silencer between the baffles by V. In fig. 1 the apertures in the baffle-plates are all axial; the theory, however, would apply if they were not axial, but were irregularly placed, so that, as in practice, exhaust gases would necessarily take a somewhat tortuous path in moving from inlet to outlet.

It is assumed, however, that the apertures in the baffle-plates and the dimensions of the inter-baffle space are small compared with the wave-length of the sound. The pressures in any particular interspace at any moment are uniform and calculable from the adiabatic volume changes which occur. Let the pressures at any moment in the

various sections of the silencer be $p_1e^{i\omega t} \ldots p_ne^{i\omega t}$ etc., and let the volume velocities in the apertures be $\dot{X}_1e^{i\omega t}$ etc.; the volume displacements will then be $(\dot{X}_1/i\omega)e^{i\omega t}$ etc.

On expressing the excess pressure in the nth section in terms of the excess of gas which has moved into it over that which has flowed out, we have

$$\frac{(\dot{X}_n-\dot{X}_{n+1})e^{i\omega t}}{i\omega V}=\frac{p_ne^{i\omega t}}{\gamma P}=\frac{p_ne^{i\omega t}}{c^2\rho},$$

where c=velocity of sound,

ρ=mean density of gas,

P=mean pressure of gas,

γ=ratio of specific heats of gas;

whence

$$(\dot{X}_n-\dot{X}_{n+1})=\frac{i\omega V}{c^2\rho}p_n. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the apertures in the baffle-plates are small compared with the wave-length of sound concerned the pressures will be uniform and the gas in each aperture will oscillate to and fro almost as though it was incompressible. Let Z_1 be the acoustical impedance (ratio of pressure to volume flow) of the air in the orifice, an impedance which arises from the effective impedance of the air to pressure exerted upon it. We then find from consideration of the motion of the air in the apertures

$(p_{n-1}-p_n)e^{i\omega t}$=pressure on air in the nth aperture

$$=Z_1\dot{X}_ne^{i\omega t},$$

$$(p_n-p_{n+1})e^{i\omega t}=Z_1\dot{X}_{n+1}e^{i\omega t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Evaluating $(\dot{X}_n-\dot{X}_{n+1})$ from considerations (2) above, and equating to the value given by (1), we find

$$(p_{n-1}-2p_n+p_{n+1})=Z_1(\dot{X}_n-\dot{X}_{n+1})=\frac{Z_1i\omega V}{c^2\rho}p_n,$$

whence

$$p_{n-1}-\left(2+\frac{i\omega VZ_1}{c^2\rho}\right)p_n+p_{n+1}=0. \quad . \quad . \quad . \quad (3)$$

Writing

$$\frac{p_{n-1}}{p_n}=\frac{p_n}{p_{n+1}} \text{ etc } =e^{\lambda}, \quad \text{say,}$$

we may rewrite (3) in the form

$$e^{-\lambda}-\left(2+\frac{i\omega V Z_1}{c^2\rho}\right)+e^{\lambda}=0,$$

whence

$$\cosh\lambda=\tfrac{1}{2}(e^{-\lambda}+e^{\lambda})=1+\tfrac{1}{2}(i\omega V Z_1/c^2\rho)=1+\tfrac{1}{2}(Z_1/Z_2), \text{ say.} \quad . \ . \ . \quad (4)$$

Now if λ is a pure imaginary, e^{λ} is a pure circular function, and p_{n+1} and p_n differ only in phase, and there is no progressive attenuation in the conduit. The condition for this is that $\cosh\lambda$ (*i. e.*, $\cos i\lambda$) shall lie between $+1$ and -1, that is, that Z_1/Z_2 shall fall within the limits

$$1>1+\tfrac{1}{2}(Z_1/Z_2)>-1. \quad . \ . \ . \ . \ . \ . \quad (5)$$

In other words the region of no attenuation is bounded by the limits

$$\frac{Z_1}{Z_2}=0\,; \quad \frac{Z_1}{Z_2}=-4. \quad . \ . \ . \ . \ . \ . \quad (6)$$

Outside these limits λ is not a pure imaginary, but is a complex number of the form $a+ib$, and attenuation occurs in the conduit.

Thc Cut-off Frequency of Baffle-plate Silencers.

If the quantity Z_1 is taken to be equal to the impedancc of an aperture of the same diameter in an infinite wall of zero thickness. we may write $Z_1=i\rho\omega/2a$, where a is the radius of the aperture.

The limits of the no attenuation region then occur at frequencies given by

$$\frac{i\omega V}{c^2\rho}\cdot\frac{i\rho_0\omega}{2a}=0 \text{ or } -4,$$

i. e.,

$$\text{when } \omega=0 \quad \text{or} \quad \text{when } \omega=2c\sqrt{\frac{2a}{V}}=2c\sqrt{\frac{K}{V}}, \quad (7)$$

where K is the well-known acoustical quantity known as the "conductivity" of the aperture.

If, however, the aperture does not consist of a single hole, but of h holes each of radius a, the conductivity is increased. If they are widely separated the conductivity becomes hK; if they are close together it probably tends to be equal to that of a single aperture of h times the area of one, and thus of radius $a\sqrt{h}$. Thus for a

silencer in which the baffle is perforated with h holes each of radius a the cut-off frequency is given by

$$\omega_c = 2c\sqrt{\frac{2am}{V}}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\sqrt{h} \gg m \gg h.$$

For a few well-separated holes m tends towards h; for a number close together the tendency towards $\sqrt{h}$ is probably operative. Rewriting this in terms of the total area S of the h perforations, we have

$$S = h.\pi a^2$$

and

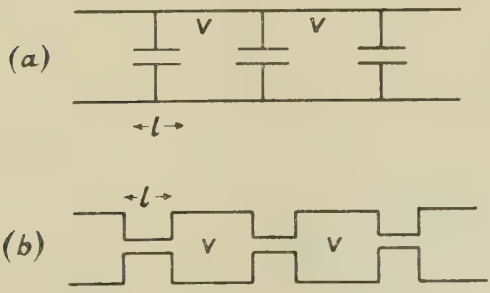
$$\omega_c = 2c\sqrt{\frac{2m}{V}}\sqrt{\frac{S}{h\pi}}. \quad . \quad . \quad . \quad . \quad (9)$$

Effect of Tubes in Baffle-plate Silencers.

If the apertures in the baffles of a silencer are of finite length and tubular (fig. 2) equation (4) still holds, but Z_1 takes approximately the value

$$i\omega\rho.\left(\frac{l}{S}+\frac{1}{c_0}\right) \quad \text{instead of} \quad \frac{i\omega\rho}{c_0}*,$$

Fig. 2.



Baffle silencer with tubes in baffle-plate.

* This follows from known impedance relations and from the fact that the conductivity of a pipe of length l is

$$\left(\frac{S}{l+\frac{S}{c_0}}\right) = \left(\frac{1}{\frac{l}{S}+\frac{1}{c_0}}\right).$$

where l is the length of the tube of cross-section S. For the cut-off frequency to be lower by an octave or more than that for a simple perforated baffle clearly $2l$ must be greater than $3\pi a$, i. *e.*, $l > 5a$ approx. Thus there is but little difference between a simple perforated baffle and one in which tubes replace the simple apertures unless the tubes are fairly long. Tubes could be curved so as to enter the chamber obliquely, and thus impart to the gases a swirling motion which might assist to cool them and thus to reduce their pressure—a factor which is not considered in the above analysis.

Limits and Application of the Theory.

When the sections of a filter are so large that they are no longer small compared with the wave-length the above theory fails. The theory may be modified, to take account of sections which have dimensions comparable with the wave-length, by following the procedure adopted by Mason (Bell Sys. Tech. J. vi. p. 258, 1927) in dealing with regular combinations of acoustical conduits. He abandoned the "lumping" of impedances and considered wave transmission to occur in each section. It should be noted, however, that if the simple theory given above fails so also does the silencer, as very high-pitched sounds pass through it. It is thus more important (as a practical point) to keep the sections small compared with the wave-length than to extend the theory to deal with other cases.

In practice also an infinite series of sections will not be used; when a finite filter is employed it is important that the inlet and outlet impedances should be the same as the characteristic impedance of the filter—that is, the same as the impedance by which the filter could be replaced for acoustical purposes.

Full calculation of the behaviour of silencers presents certain difficulties on the acoustical side on account of the fact that silencers do not conform to ideal sizes nor to ideal geometrical forms; moreover, no allowance has been made above for the progressive cooling of exhaust gases as they pass through the conduit, nor for the fact that the large displacements which occur in an actual silencer may lie outside the limits to which pressures may be assumed to be proportional to displacements.

It may be mentioned however that in a certain perforated baffle silencer employed in association with oil engines the baffle aperture is (731 sq. in.) 4700 sq. cm. The interbaffle volume is (5 ft. and 2 ft. long) 1,100,000 c.c. On the assumption that the outlet is a single hole the cut-off frequency calculated from formula (9) ($m=h=1$) is 90 cycles per second ($\omega=2\pi n=1760$). On the other hand, if the outlet is not a single hole, but several holes having a total area of 731 sq. in., the cut-off frequency may be 90 cycles per second or rather higher Another silencer has an interbaffle volume of 1,350,000 c.c. and baffles each with 136 holes of total area about 2700 sq. cm. From formula (9) the cut-off would be in the region of 69–240 cycles per second, the lower figure being the closest approximation.

Unfortunately at present there are no experimental data for comparing with the above theory, but the calculations indicate that baffle-type silencers act as low-pass filters and tend to suppress all but very low-pitched sounds—a result in accordance with general impressions when the noises of unsilenced and silenced exhausts are compared.

LXX. *The Structure of Thin Celluloid Films.*—I. *By* E. Taylor Jones, *D.Sc., Professor of Natural Philosophy in the University of Glasgow* *.

[Plate XXIII.]

A NUMBER of electron diffraction photographs, obtained with thin films of celluloid or nitrocellulose, have been described by the writer †, in which the pattern appeared to be double—that is, in which each circular ring of the pattern had two sets of six uniformly spaced spots on its circumference, the two groups having apparently no definite connexion with each other. The two sets might appear in the photograph as twelve uniformly spaced spots on the circumference, or they might form six pairs, or double spots,

* Communicated by the Author.

† Phil. Mag. xii. p. 644 (1931); 'Induction Coil Theory and Applications,' p. 158 (Pitman, 1932).

the angular separation of a pair being considerably less than the angle between the pairs.

The photographs were obtained by the transmission method (first used by Professor G. P. Thomson), and it was suggested that the two superposed patterns were produced in neighbouring portions of the film, or in portions near its two surfaces, where the scattering centres were arranged in lattices which had the same spacing constants but which were set at an angle to each other about a normal to the film.

In the present communication a photograph will be described in which the pattern appears to be *quadruple*, being composed of two double patterns. Each double pattern consists, it is suggested, of two forms having, not circular, but elliptical rings, the major axes of the ellipses being inclined to each other. One of the two double patterns is much more intense than the other, and this will therefore be referred to as the main or principal pattern. The other, much weaker in intensity but of larger dimensions, indicating smaller spacing constants, will be called the secondary pattern.

Most of the spots in the photograph are formed by a single scattering of incident cathode rays, but a number of those in the secondary pattern appear to be produced from rays already diffracted in one layer by a second diffraction in another layer.

Two copies of the photograph, both prepared from the same negative, are shown in figs. 1 and 2 (Pl. XXIII.). Owing to the great difference of density in different parts of the plate it is difficult to show the whole of the pattern in one print, and the print of fig. 1 (Pl. XXIII.) was therefore exposed so as to show the details of the central portion of the pattern, that of fig. 2 (Pl. XXIII.) to show more of its outlying parts.

The discharge-tube and vacuum camera employed in taking the photograph were similar to those previously described, but larger, the distance from film to plate being 36·0 cm. The same method of experimenting also was adopted, the potential being generated by a coil and limited by a spark between spherical electrodes passing simultaneously with the discharge.

It will be seen that, owing to the heterogeneity of the rays produced while the potential is rising to the sparking value, the spots in the photograph are not circular,

but form lines or streamers radiating from the centre of the pattern. The distance of a spot from the one diametrically opposite to it was taken as the shortest distance between the two streamers with their width added to it. This should give the distance between the centres of the circular "heads" of the streamers, and should be the ring diameter corresponding to the sparking potential.

All the prominent diffracted spots in figs. 1 and 2 (Pl. XXIII.) belong to the principal pattern. It will be seen that they all lie in equidistant horizontal rows. The central row has a number of single spots, the first upper and lower rows a number of pairs of spots. The second upper and lower rows also consist of pairs, but in these rows the spots are more nearly uniformly spaced. There are also a few faint spots in third upper and lower rows.

So far as the secondary pattern can be observed it consists of horizontal rows which lie midway between the rows of the main pattern. A few of the secondary spots are visible in the photograph of fig. 2 (Pl. XXIII.), the most conspicuous being those between the central and first and between the first and second principal rows.

We will consider the details of the principal and secondary patterns separately.

The Principal Pattern.

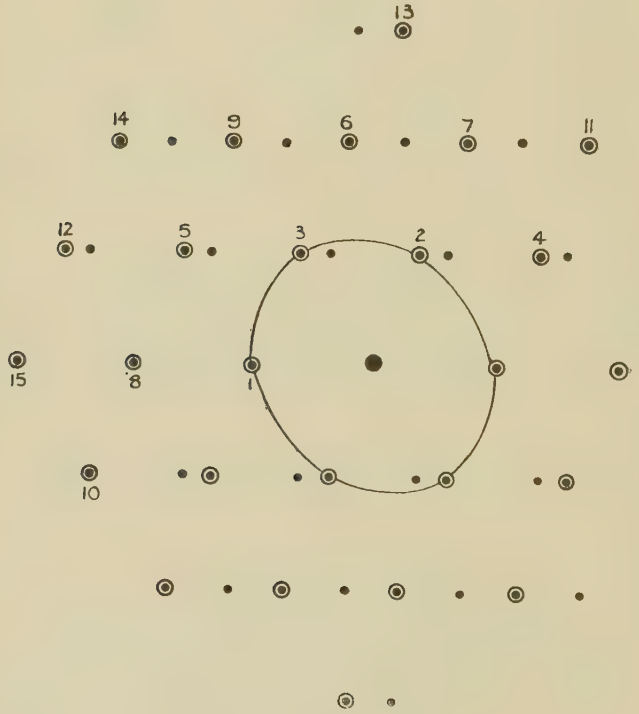
In fig. 3 are shown the positions of the "heads" of the streamers forming the principal pattern. The six spots which are nearest the centre lie on a circle of diameter 10·3 mm. in the photograph, but they are not uniformly spaced on its circumference. Consequently it is necessary to divide the pattern into two groups of spots, each of which forms an "elliptical" pattern *. The two groups will be called the first and second principal patterns, the first, marked ⊙ in fig. 3, being slightly the more intense in the photograph.

In fig. 3 is also indicated the first (elliptical) ring of the first principal pattern. The ratio of the axes of this

* It is not supposed that the film has a fibrous structure which would cause it to give a "rotation" photograph. See Sir William Bragg, 'An Introduction to Crystal Analysis,' p. 141 (Bell, 1928).

ellipse is 1·17 and its diameters at the points marked 1 and 2 are equal. The first elliptical ring of the second principal pattern is symmetrical with the ellipse shown in fig. 3 about a vertical (or horizontal) axis through the centre of the figure, and the angle between the major axes of the two ellipses is 66° 42′. The point

Fig. 3



The principal pattern.

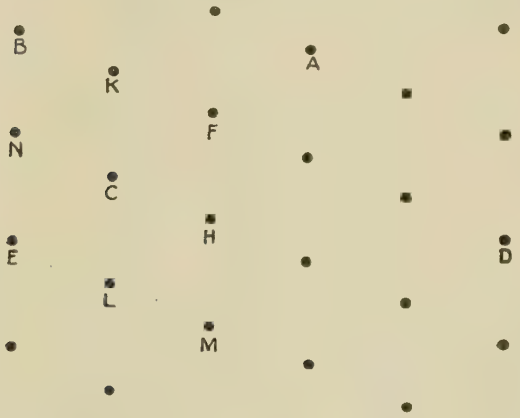
marked 1 and its opposite are common to the two ellipses.

The Principal Lattices.

The principal and secondary patterns are formed by certain arrangements of scattering centres which we will call the principal and secondary lattices, leaving open for the moment the question of the relation between

them. A simple arrangement, by which the first principal pattern could with considerable accuracy be produced is indicated in fig. 4, which shows the projection of this "first principal lattice" upon a plane parallel to the surfaces of the film. It consists of an array of points in triangular order, the vertical rows parallel to KL and those parallel to BD having the greatest spacing and the greatest number of points per centimetre. There are two sets of rows, parallel to BM and AE respectively, which are at right angles to each other. They are nearly parallel to the principal axes of the

Fig. 4.



Projection of first principal lattice.

ellipse shown in fig. 3, and they are axes about which the other rows of points in fig. 4 are symmetrical. The spacings of rows parallel to AE and BM are in the ratio 3/2, their values being 0·9015 and 0·601 of the greatest spacing—that is, of the spacing of rows parallel to KL or BD.

The second principal pattern is formed by a lattice which is symmetrical with that of fig. 4, and may be seen in position by turning fig. 4 clockwise through about 67° in its own plane.

The positions of all the diffracted spots in the principal pattern (fig. 3) can be accounted for by these two lattices, that is, by reflexion from lattice planes which are normal

to the surfaces of the film and parallel to the various rows of points represented in fig. 3 and its counterpart. For example, the spots numbered 1 and 2 in fig. 3, and their opposites, are formed by lattice planes parallel to KL and BD (fig. 4) respectively, and are equidistant from the centre. No. 3 and its opposite are formed by planes parallel to AE, no. 4 by planes BM, nos. 8 and 7 are the second orders of nos. 1 and 2, and no. 15 the third order of no. 1.

First Principal Pattern.

No. of spot.	$\frac{D}{D_1}$.	Lattice plane.	Spacing ratio.	Spacing ratio $\times \frac{D}{D_1}$.
1	1	KL	1	1
2	1	BD	1	1
3	1·11	AE	0·9015	1
4	1·66	BM	0·601	0·998
5	1·86	AH	0·5376	1
6	1·86	AK	0·5376	1
7	2	BD_2	1	2
8	2	KL_2	1	2
9	2·23	AE_2	0·9015	2·01
*10	2·56	BL	0·3912	1
*11	2·56	BH	0·3912	1
*12	2·81	AM	0·361	1·014
13	2·81	AB	0·361	1·014
*14	2·87	AL	0·3453	0·991
*15	2·95	KL_3	1	2·95

Spots due to symmetrical sets of planes are bracketed.
The suffixes in BD_2 etc. indicate the order of reflexion.

* Measured from the centre, the opposite spot being very faint or invisible.

The Table shows in the second column the ratio of the diametral distance D of each spot of the first principal pattern to that (D_1) of the spot no. 1, the spots being numbered (first column) as in fig. 3. The third column gives the lattice planes (fig. 4) which are responsible for the spot, the fourth column the ratio of the spacing to that of the planes KL or BD calculated from fig. 4. The fifth column shows the product of D/D_1 and the spacing ratio.

It will be seen that when allowance is made for the "order" of the reflexion the numbers in the last column are practically all equal to unity, indicating that the relative distances of the spots from the centre are correctly given by the spacing constants of the lattice of fig. 4. The directions of the spots with respect to the centre are also correctly given, and it is clear that the positions of the spots of the second principal pattern are equally well accounted for by the lattice symmetrical with that of fig. 4. It may be concluded that the whole of the principal pattern can be explained by reflexions from net-planes perpendicular to the plane of fig. 4 and its counterpart. These reflecting planes must make small angles with the incident pencil *, and we assume that they are normal to the surfaces of the film, the necessary small glancing angles being provided by the curvature of the film.

There is evidence to show that the curvature of a film plays an important part in the formation of a pattern. For example, a certain gold film examined by the writer gave a pattern in which the first ring showed only two maxima instead of the usual four—that is, the (2 0 0) spots appeared, but not the (0 2 0) spots. Examined under a lens the film showed a distinct "furrow" across the centre in a direction at right angles to the line joining the (2 0 0) spots, the inference being that the normal section of the film parallel to the length of the furrow possessed insufficient curvature to admit of even a first order reflexion by normal planes at right angles to this direction. The fact that the photograph of fig. 2 (Pl. XXIII.) shows more of the distant spots on the left-hand side than on the right may be due to a dissymmetry of the curvature of the film, *i. e.*, to the centre of curvature being slightly to one side of the incident pencil.

According to the view here suggested of the manner of formation of the principal pattern, viz., that all its spots are due to reflexions by normal planes of the film, it is clear that no information is afforded by the form of this pattern as to the spacing of net-planes which are parallel or inclined to the surfaces of the film. The only condition required, so far as the principal pattern alone is concerned, is that all the scattering

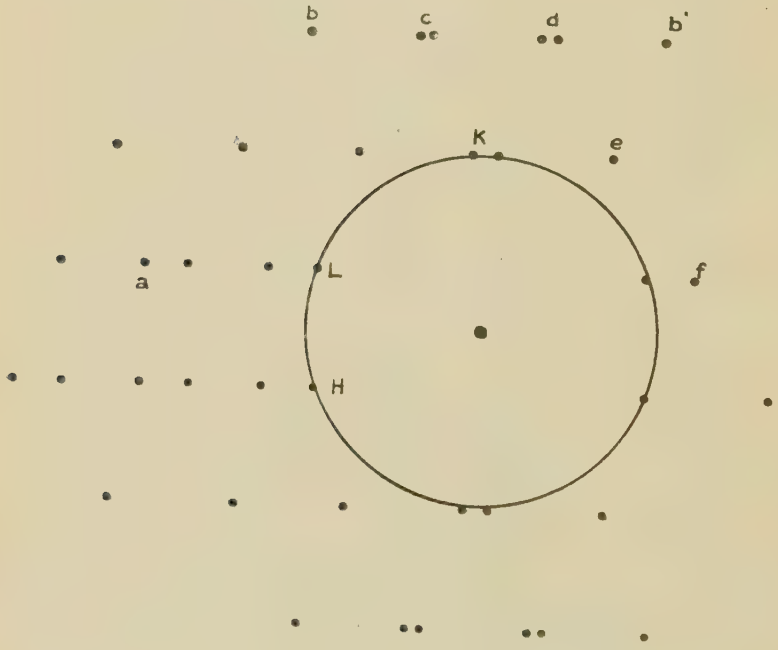
* The angle between the diffracted pencil forming spot no. 1 (fig. 3) and the incident pencil is less than 1°.

centres of the principal lattices must lie in columns at right angles to the film standing upon the points marked in fig. 4 and the figure symmetrical with it.

The Secondary Pattern.

The spots of the secondary pattern are shown in fig. 5, and, so far as they appear in the photograph, they lie,

Fig. 5.



The secondary pattern.

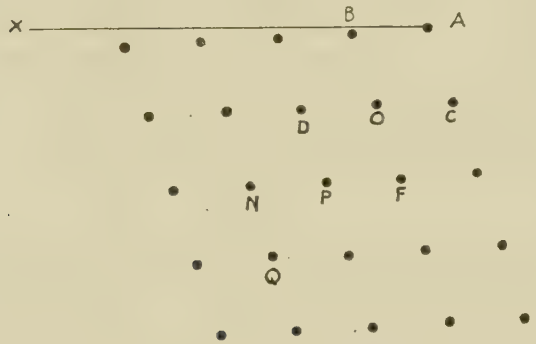
as stated, in rows midway between those of the main pattern. The spots in fig. 5 which are nearest the centre, *i. e.*, H, L, K and their opposites, lie on a circle of diameter 14·2 mm. in the photograph * ; but again they are not spaced uniformly on the circumference.

* The measured ratio of this diameter to that of the "first circular ring" of the main pattern is therefore 1·38. The condition that the spots H, L, K and their opposites may fall exactly midway between rows of the main pattern is that the ratio of these diameters should be 18/13.

This fact, and the fact that K is double, suggest that fig. 5 is a double pattern somewhat similar to the main pattern but of different dimensions.

Of the spots H, L, K in the photograph H and the left-hand member of K are rather stronger than the others, and we therefore assume that they belong to the same group, which we shall call the first secondary pattern. They lie on the (elliptical) first ring of this pattern, the diameters of which at H and K are equal. The spots at the ends of the major axis of this ellipse are not apparent in the photograph: if they are formed at all they probably fall very near certain spots of the main pattern (the companion of no. 5, fig. 3, and its opposite) by which they are masked.

Fig. 6.



Projection of first secondary lattice.

Again assuming reflexion by normal planes, we find that a lattice which might account for the formation of the first secondary pattern is indicated in fig. 6, which represents the projection of the lattice upon a plane parallel to the (horizontal) surfaces of the film. The scattering centres lie at the angles of a series of triangular prisms standing vertically upon triangles such as OAC, the angles of which are OCA = 73° 22′, and AOC and OAC, each equal to 53° 19′. The greatest spacing (*b*) of planes in the lattice is that of the planes parallel to AC or AB, these planes being responsible for the spots H, the left-hand member of K (fig. 5), and their opposites. There are again two planes of symmetry, parallel and perpendicular to OA, the spacing

of planes parallel to OA being 0·837 b and of those parallel to BC 0·623 b. This lattice and the lattice symmetrical with it (shown by turning fig. 6 counter-clockwise through 67° 42′ in its own plane *) can account for the eight spots on the circle shown in fig. 5 ; but the only other spots in fig. 5 for which they could account by a single reflexion of primary rays are those marked a, b, b', and the corresponding spots in the lower half of the diagram, which lie very near the positions indicated by reflexions from the planes AD, AF (fig. 6) and the corresponding planes of the symmetrical lattice.

It is, however, improbable that spots so far from the centre as a, b, b' (fig. 5) are formed in this way. In weak photographs it is usually only the spots nearest the centre which appear, and even in photographs of considerable intensity the diffracted rays are sometimes practically confined to those forming the first ring. It is probable therefore that in the weak secondary pattern here under consideration the only spots which are formed directly by a single diffraction from the primary pencil are the eight H, L, K, and their opposites which lie on the circle in fig. 5. The large majority must be accounted for in some other way.

Now an examination of the photograph or of figs. 3 and 5 shows that each spot of the secondary pattern outside the circle in fig. 5 is placed with reference to one in the main pattern in the same way as one of the spots H, L, K is situated in relation to the centre. This suggests that most of the spots of the secondary pattern are formed by two diffractions, the rays passing through two layers of the film in succession and being diffracted in both †.

For example, of the row of six rather prominent spots running to the left from H (fig. 5) the first (H) is formed from the primary pencil, the second from the pencil to the opposite of no. 2 (fig. 3) just as L is formed from the primary pencil, the third from the no. 1 ray, the fourth from the opposite of no. 4, the fifth from no. 8, and the sixth from no. 10. Of those running to the left from L

* The angle between the line AB (fig. 6) and the horizontal axis Ax in the figure is 2° 50′.

† The " streamers " of rays diffracted twice would still point radially from the centre.

the second is derived from the ray to the companion of no. 3 as H is from the primary pencil, the third from no. 1, the fourth (*a*) from the companion of 5, and the fifth from no. 7. Again, of the three spots to the left of K (fig. 5) the first may be derived from the no. 1 ray as one component of K is from the primary pencil, the second from the no. 3 ray as L is from the primary pencil, and the third in the same way from no. 5. The spot *b* is derived from the ray to the companion of 5 as the left-hand component of K is from the original pencil. As to the double spot *c* (fig. 5) its left-hand member appears to be formed from no. 3 as the right-hand K spot is from the primary pencil; the right-hand member of *c* from the companion of 3 as the left-hand K is from the original pencil. This accounts correctly for the diminished separation of the two components of *c* *. The double spot *d* is formed in a similar way from the pair no. 2 and its companion, *e* from the opposite of no. 1, and *f* from 2.

In this way all the spots in fig. 5 can be accounted for, it being assumed that the curvature of the film is sufficient to allow pencils already reflected by certain planes in one layer to be reflected again by planes (not parallel to the first) in a subsequent layer of the film.

It may be noticed that diffracted rays forming the *first* principal pattern can only be diffracted again to form spots of the *second* secondary pattern; and, similarly, *second* principal rays can give rise only to *first* secondary rays. As a rule the latter spots are weaker, in some cases much weaker, than the former.

Instead of being secondary rays formed by a second reflexion of principal diffracted rays the twice-reflected rays may equally well be regarded as "principal" derived from "secondary" rays. The photograph apparently provides no evidence as to the order of succession of the reflexions.

The obvious inference from these relations is that the principal patterns are formed in two adjacent portions of the film, in which the lattices are similar but set at an angle of about 67° to each other, the primary pencil

* In the photograph the separation of the two spots no. 3 and its companion (fig. 3) is 1·15 mm., that of the spots K (fig. 5) 0·7 mm., and that of *c*, 0·45 mm.

falling partly on each portion. Above or below each of these portions is a thinner layer, of different lattice dimensions in the plane of the film, the lattices of the two thinner layers being also similar to each other. It is remarkable that the angle between the two secondary lattices is nearly the same as that between the principal lattices, the *minor* axis of symmetry in a secondary lattice (*e. g.*, OA in fig. 6) being, however, parallel to the *major* axis of symmetry of the material which lies above or below it, *i. e.*, the line corresponding to BM in the counterpart of fig. 4.

These relations must arise in the manner in which the groups of atoms become packed in the film when it is deposited from an evaporating solution. The relation between the principal and secondary lattices can be better considered when we know their absolute dimensions.

The Dimensions of the Lattices.

The spacing constants are best determined by comparison of the dimensions of the pattern with that given by a gold film, the greatest spacing in which, *i. e.*, that of the (2 0 0) planes, is known to be 2·032 Å.U. If this distance be denoted by d_0, and if D_0 is the diameter of the first ring of the gold pattern, we have, for wavelength λ, from the Bragg relation,

$$D_0 d_0 = 2l\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where l is the distance from film to plate.

If it be assumed that at the same voltage the wavelength in the celluloid film is the same as that in the gold film it follows that if D_1 is the diameter of the first circular ring in the celluloid principal pattern and a the corresponding spacing,

$$a = \frac{D_0 d_0}{D_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The diameters D_0 and D_1 were determined for a number of voltages ranging from 40 to 80 kv., the mean value of the ratio D_0/D_1 being found to be 2·015. Hence by (2) the greatest spacing (a) of normal planes in the thicker layers of the celluloid film, *i. e.*, that of the planes KL or BD (fig. 4) is 4·09 Å.U. For the planes of symmetry in these layers the spacings are 2·46 Å.U.

for normal planes parallel to BM and 3·68 Å.U. for those parallel to AE (fig. 4).

In the thinner layers the greatest spacing of normal planes, from the ratio of the diameters of the first rings in the secondary and principal patterns, is $b = 2{\cdot}97$ Å.U. This is the spacing of planes parallel to AC or to AB (fig. 6). For the planes of symmetry the spacings are therefore 1·85 Å.U. for the planes BC and 2·49 Å.U. for the planes OA (fig. 6).

There is therefore a simple relation between the two lattices represented in figs. 4 and 6. Considering distances along and at right angles to the axes of symmetry, we find that the distance EC in fig. 4 is practically equal to BC in fig. 6 and that KH (fig. 4) is almost exactly twice the distance OA in fig. 6. The differences are so small that we shall assume these relations to be exact, and we shall also assume that, as already indicated, the normal planes containing the major axes of symmetry in the principal layers are parallel to those containing the minor axes in the thinner layers.

We now find that the counterpart of fig. 6 can be derived from fig. 4 by turning the latter through 60° about the line AE and then projecting it upon the plane of the former. In other words, by turning fig. 4 (which refers to one of the thicker layers) through 60° about AE and projecting it upon a plane parallel to the surfaces of the film we obtain the corresponding figure for the thinner layer which lies above or below it.

This relation between figs. 4 and 6 and their counterparts suggests that these figures are projections of the same lattice turned into four different positions, and that therefore only one lattice is concerned in the formation of the four patterns.

But it is not possible to determine the lattice completely from these projections. There are many lattices which satisfy the conditions of having projections such as those of figs. 4 and 6 on different planes. We choose, however, the simplest solution, viz., that the spots in fig. 4 represent actual lattice points, and that parallel to this plane above and below are other similar net-planes placed at regular intervals. On this assumption it can be shown that if the spacing of planes parallel to that of fig. 4 is 4·256 Å.U., and if the points B, C

(fig. 6) are placed vertically below the points C, F (fig. 4), all the scattering centres of this lattice, those in the plane of fig. 4 and those above and below it, would, after a rotation of 60° about AE, fall into the vertical lines standing upon the points marked in fig. 6. For example, the point next above H (fig. 4) falls on rotation into the column on P (fig. 6), the second above H into the column on Q, the first above K into the column on O, the first above M into the column on N, and so on.

Accordingly we conclude that the whole of the pattern can be explained as being formed by a single space-lattice in different positions, and it remains to consider the nature of its lattice points. It is generally agreed that the molecule of native cellulose consists of a chain of 6-atom rings * each of which occupies, according to X-ray measurements, a length of 5·15 Å.U. along the chain. According to Meyer and Mark † the width occupied by a ring and its attachments is 8·25 Å.U. and the spacing of the parallel planes containing the rings is 3·93 Å.U. Astbury and Marwick ‡ have suggested, from an examination of the data for a number of allied substances, that on the average the space occupied by one ring has a length of about 5½ Å.U. along the chain, a width of about 7½ Å.U., and a depth at right angles to the plane of the ring of about 4½ Å.U. From his study of monomolecular films N. K. Adam § has concluded that in a cellulose derivative the rings at low pressures lie flat on the water, the area occupied by each being about 55 to 60 sq. Å.U. at 2 dynes per cm.

By X-ray measurements of a series of nitrocelluloses of different nitrogen content F. D. Miles || has found that the spacings increase slightly as the nitrogen increases from 10 to 12·8 per cent., and at this point increase suddenly by an amount which may be as great as 15 per cent.

Now the positions of the points marked in fig. 4 need not be all occupied, and by omitting some of them the figure can be made to appear as a net of hexagonal rings

* See Haworth, 'The Constitution of Sugars' (Arnold, 1929).

† 'Der Aufbau der Hochpolymeren Organischen Naturstoffe' (Leipzig, 1930).

‡ 'Nature,' cxxvii. p. 12 (1931).

§ Trans. Faraday Soc. xxix. 1, p. 100 (1933).

|| Trans. Faraday Soc. xxix. 1, p. 117 (1933).

such as FKNELH *. These are the smallest hexagons into which the figure can be divided. The width KH (7·36 Å.U.) and the depth from layer to layer (4·256 Å.U.) agree fairly well with the width and depth of a 6-atom ring according to some of the above data, but the length FE (9·9 Å.U.) is about twice the usually accepted value for the length of a ring along the chain. So far as the writer is aware there is no other evidence of such abnormal lengthening of the chains, and we must conclude that the points F, K, N, etc. in fig. 4 cannot represent the individual atoms forming one of the rings.

Apparently the only way in which the length can be reduced to the accepted value is to suppose that each point in fig. 4 represents not a single atom but one of the rings. Such group centres, if arranged at regular intervals, should produce much the same diffraction effects as single centres arranged in the same way. We suppose therefore that the chains lie along AE and parallel lines in fig. 4, each point in these lines representing the centre of a ring. This would give 4·95 Å.U. for the length of a ring, and if every row such as AE is occupied $\frac{1}{2}$KH, *i. e.*, 3·68 Å.U. for the width. The area is therefore much too small, and in order to bring it up to the order stated by N. K. Adam we must suppose that only one row, such as AE, in every three in the plane of fig. 4 is occupied by a chain.

The arrangement arrived at is therefore as follows :— In the plane of fig. 4 the row KN is occupied by a chain, but AE and HL are vacant. In the plane above that of fig. 4 the row above KN is vacant, that above AE is occupied, that above HL vacant. In the next plane above the rows over KN and AE are vacant, that above HL occupied. In the fourth plane the row above KN is occupied, and so on. With this arrangement the normal planes and their spacings are preserved, and the pattern would be that of the photograph of figs. 1 and 2 (Pl. XXIII.).

The volume of the space occupied by a ring is $4{\cdot}95 \times 11{\cdot}04 \times 4{\cdot}256$, *i. e.*, 233 cubic Å.U. If the composition is $C_6H_9O_4$ (NO_3) and the mass of a hydrogen atom is $1{\cdot}662.10^{-24}$ gm., the mass of a ring is 344.10^{-24} gm.

* This kind of arrangement was suggested by the writer on a former occasion (Phil. Mag. xii. p. 656, 1931), but the system of group centres suggested in the present paper agrees better with X-ray measurements.

The density of the film is therefore 1·476 gm./c.c. If the composition is $C_6H_8O_3(NO_3)_2$ the mass of a ring is $418 \cdot 10^{-24}$ gm. and the density 1·8 gm./c.c. *.

The density of the celluloid sheet with a portion of which the solution was prepared is 1·330 gm./c.c., this small value being doubtless due to the camphor which the sheet contains. It is improbable that the film contains any camphor, at least in the portions through which transmission took place. Photographs showing a first ring of about the same diameter as that of figs. 1 and 2 (Pl. XXIII.), indicating a maximum spacing of about 4·1 Å.U., have been obtained by the writer with films of pure nitrocellulose †. According to W. G. Burgers ‡ the principal spacings in a camphor crystal (determined by X-ray methods) are about 5 or 6 Å.U., and no spacings as large as this have been observed by the writer in celluloid films §.

It should be added that in addition to the pattern already described the photograph shows another diffraction effect in the form of three concentric circular rings, all rather broad and diffuse and uniform in intensity round the cicumference. The diameter of the first of these rings is about the same as that of the "first circular ring" of fig. 3, that of the second nearly twice as great, and that of the third $2\sqrt{3}$ times the first. It is possible that the second ring consists of two (diameters $\sqrt{3}\,D_1$ and $2D_1$) which appear merged into one. These diameter ratios are among those discussed in the previous communications referred to, and the rings probably arise in a part of the film where there are many small "crystals" turned into various positions about the normal to the film.

Natural Philosophy Department,
The University, Glasgow.
May 1933.

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* If every second row in fig. 4 and parallel planes were occupied the width of a ring would be 7·36 Å.U., and the density would be for the lower nitrate 2·22, for the other 2·7 gm./c.c.

† 'Induction Coil Theory and Applications,' pp. 175, 241.

‡ 'Nature,' cxviii. p. 116 (1926).

§ N. K. Adam states (Trans. Faraday Soc. xxix. 1, p. 103, 1933) that in monomolecular films the camphor goes completely into solution in the water.

LXXI. *On Models of the Electric Field and of the Photon.*

By Sir. J. J. Thomson, *O.M., F.R.S.**

Summary.

The model discussed in this paper is one where the ether is regarded as a gas whose molecules, called granules, are so small as to be almost infinitesimal even in comparison with an electron. The granules move so fast that the velocity of sound through the gas is equal to the velocity of light. The granules are regarded as the source of all mass and energy, the mass of any portion of matter and its energy being proportional to the number of granules associated with it. Matter is supposed to consist of electrons and protons which carry charges of electricity and are the ends of tubes of electric force. Tubes of force are represented in the model by vortex filaments in the fluid formed by the granules. Such filaments are shown to have all the properties postulated for tubes of force.

When a vortex filament moves it carries along with it a portion of the surrounding fluid; thus its effective mass is greater than the mass of the granules in the filament itself. The amount of fluid bound to a filament depends upon external conditions, so that the effective mass of a filament is not constant but depends upon its surroundings. In the model the mass of a body is represented by the effective mass of the filaments to which it is attached.

It is shown that the model leads to Maxwell's equations of the electric field, so that it can illustrate any phenomenon which can be deduced from these equations.

The equations given by the model are equations between average values; they, like the equations in the Kinetic Theory of Gases, are statistical. Maxwell's equations do not lead to the "Photon." This in the model appears as a closed vortex ring in the fluid formed by the granules; the ring travels through the fluid with the velocity of light, and is accompanied by a train of waves in the fluid, the frequency of the waves being that of the photon. When the photon is travelling through space devoid of matter it is moving in the direction of

* Communicated by the Author.

propagation of the waves, but when this direction is changed, *e. g.*, by diffraction at a grating, and is no longer in the direction of motion of the photon, it is shown that a couple acts on the photon, tending to make it move in the direction of propagation of the waves.

The model also indicates that a moving electron would be accompanied by a system of waves, and gives the correct expression for the wave-length.

THE object of this paper is to describe a model in which the phenomena of the electric field find a parallel in mechanical processes going on in the model. It is sometimes objected that the discoveries of modern physics are so contrary to the laws of ordinary dynamics that it is hopeless to attempt to represent them by a model. This paper is an attempt to show that it is not so. It would indeed be surprising if it were, because some of these discoveries, *e. g.*, the effect of electric charge and velocity on mass, were deduced by the old dynamics and Maxwell's equations before the advent of relativity. In the model I propose the ether is regarded as molecular in structure, and to consist of exceedingly small particles called granules, all having the same mass and moving with the same velocity. The mass of a granule is so small, almost infinitesimal in comparison even with that of an electron, that the velocity of sound through this molecular ether is equal to the velocity of light.

The granules are regarded as the source of all the mass and energy possessed by matter, and the mass of a portion of matter is the sum of the masses of the granules attached to it, and its energy the sum of their energies. Thus mass and energy are proportional to each other, and an increase in energy makes a proportional one in mass. Mass and energy can only be detected or measured when associated with matter, so that the mass of a free granule is not in reality mass at all, but only the potentiality of becoming mass when the granule is bound to matter.

An integral part of matter consists of electrons and protons, these being electrified are the ends of tubes of electric force; it is these tubes which in our view bind the granules to matter. The properties possessed by tubes of electric force, *e. g.*, that they can neither be

created nor destroyed and that they must either form closed curves or have their ends on charges of electricity, are paralleled by the properties of vortex filaments in a fluid; these, too, must form closed curves or else terminate on a boundary of the fluid in which they are moving. We may picture the granules as the molecules of a fluid in which the tubes of force are vortex filaments in the fluid, while electric charges correspond to boundaries of the fluid. We shall first proceed to show that these filaments possess the properties which Faraday postulated for his tubes of force, *i. e.*, that they are in a state of tension and that the tubes when at rest repel each other.

Tension in the Tubes.

Consider a circular cylinder containing vortex filaments parallel to its axis uniformly distributed over its cross-section. Let p be the pressure at a distance r from the axis, v the tangential velocity of the fluid at this distance. Since the circulation round a circle of radius r is 4π times the vorticity inside the circle,

$$2\pi rv = 4\pi\sigma m \times \pi r^2,$$

where σ is the number of vortex filaments passing through unit area and m the strength of each filament. Thus

$$v = 2\pi\sigma m r\,;$$

and if ρ is the density of the fluid

$$\frac{dp}{dr} = \rho\frac{v^2}{r} = 4\pi^2\sigma^2\rho m^2 r\,;$$

if Π is the pressure at the outside of the cylinder where $r = a$,

$$p = \Pi - 2\pi^2\rho\sigma^2 m^2(a^2 - r^2).$$

The pressure exerted over a cross-section of the cylinder is

$$\int_0^a p 2\pi r dr = \Pi\pi a^2 - \pi^3\rho\sigma^2 m^2 a^4.$$

Thus the pressure is less than if there were no vorticity by $\pi^3\rho\sigma^2 m^2 a^4$.

If this cylindrical bundle of vortex filaments were to end on the surface of an electron the pressure on the electron would be reduced by this amount, and thus we may regard the bundles as exerting a tension equal to

$$\pi^3\rho\sigma^2 m^2 a^4.$$

The energy per unit length of the bundle is

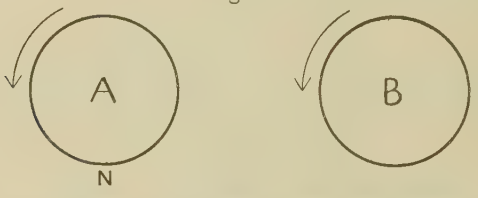
$$\tfrac{1}{2}\int_0^a \rho v^2 2\pi r dr = \pi^3 \rho \sigma^2 m^2 a^4,$$

and is thus equal to the tension in the bundle. Thus the filament exerts a tension per unit area equal to the energy per unit volume of the fluid.

Repulsion of Vortex Filaments.

Consider a very simple case, that of two straight parallel filaments rotating in the same sense and held fixed. Let A and B be the cross-sections of the filaments, their axes being at right angles to the plane of the paper. Then the rotation due to the filament B will produce on the left-hand side of A a velocity in the same direction as that due to A itself, while on the right-hand side of A the velocity due to B will be in the opposite direction to that due to A. The velocity of the fluid on the left-hand side of A will be greater than that on the right, and

Fig. 1.

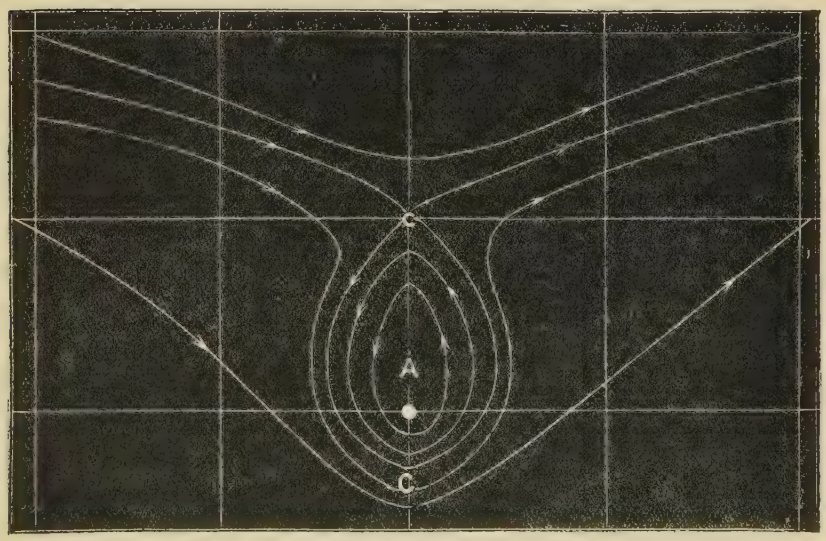


therefore, by Bernoulli's principle, the pressure on the left will be less than that on the right, so that A will be repelled from B. This is a special case of the general theorem (Lamb's 'Hydrodynamics,' p. 183, 5th edit.) that the mechanical forces between vortex filaments at rest are equal in magnitude but *opposite* in sign to those which would exist between conducting wires coincident with the filaments and carrying electric currents proportional to the strength of the filaments. As two currents in the same direction attract each other, two vortex filaments of the same sign will repel, and if the filaments are parallel the force per unit length will vary inversely as the distance between them. If the filaments are moving instead of being held fast the forces will be modified by the introduction of those acting on a rotating body when moving through a liquid which make it try to follow its nose. Thus, for example, if in fig. 1

A were to move downwards, as it would under the action of B if it were not held, the new force would be one in the direction in which its nose N is moving, *i. e.*, tending to make A move towards B. This is in the opposite direction to the force when A was fixed and in the same direction as the force in the electrical analogue.

A point of great importance in this representation of the electric field is that a vortex filament, when drifting along in a stream in the fluid, has attached to it and carries along with it a portion of the liquid in which it is

Fig. 2.



moving. Let us take the simple case of a long straight filament whose cross-section is at A and whose axis is at right angles to the plane of the paper; the rotation in the filament is in the direction of the arrow and the filament is supposed to be moving from left to right. The lines of flow of the fluid relative to the filament are represented in the figure and are seen to be of two kinds: (1) those inside the curve which cuts the vertical line through A in the points CC (these form closed curves surrounding the filament); (2) those which are outside this curve and go off to infinity in both directions. The fluid between the filament and CC is moving irrotationally,

but since CC is a line of flow no fluid crosses it; all the fluid inside CC moves along with the filament; the mass and energy of this fluid is thus bound to the filament and may be regarded as possessed by it. The liquid outside CC has also mass and energy, but it is not in any way permanently bound to the filament; it flows past it, and a portion which at one time is quite close to the filament may shortly afterwards be at a great distance from it. Considered with relation to the filament the mass and energy of the fluid outside the curve CC may be called free mass and free energy.

If the filament is moving in a closed vessel the shape of CC and the amount of fluid enclosed by it will alter with the distance of the filament from the boundaries of the vessel or from other solids which may be in the vessel, and the mass and energy of the filament will alter too. This is an example of how mass may vary with the external conditions. The energy and mass of the fluid as a whole remains constant, and the mass or energy lost or gained by the filament and its attachment inside CC will be gained or lost by the free mass outside.

The circular vortex ring is another good example of the difference between the mass of the rotationally moving fluid and the effective mass of the ring.

The mass of the rotationally moving fluid is $\rho 2\pi a . \pi e^2$, where a is the radius of the circular axis of the ring and e that of its cross-section.

The effective mass when a/e is large is

$$\rho 4\pi^2 a^3 / \log \frac{8a}{e};$$

the ratio of this to the former mass is

$$2(a^2/e^2)/\log \frac{8a}{e},$$

which is very large when a/e is large.

The granules, being particles of a fluid in which there are discrete vortex filaments, may thus be divided into two classes:

1. Those which are attached to vortex filaments and are carried along by them. We shall call these bound granules, and the mass and energy they possess bound mass and bound energy.

2. Those which are not carried along by the vortex filaments and have no lasting connexion with them. We shall call these free granules, and their mass and energy free mass and free energy.

The number of free granules in any volume is in general enormously greater than that of the bound.

The filaments associated with bound mass or energy must end either on electrical charges, *i. e.*, electrons or protons, or else form closed curves. In the former case the mass of the granules with which they are associated becomes part of the mass of the electron or proton, and hence a part of the mass of an atom which contains them. If the vortex filament forms a closed curve we get something which is intermediate between ether and matter and which has many of the properties of the photon—it has mass and energy, but is not charged with electricity.

All bound energy is due to the motion of granules, and is thus kinetic energy. For example, what is usually called the potential energy of an electron and expressed by $e^2/2a$, where a is the radius of the electron, is on this view the kinetic energy of granules rotating round and bound by vortex filaments coming from the electron. The energy of the free granules in the ether is not really energy, but only the potentiality of becoming energy when linked to matter.

Though free granules cannot give up energy to matter and remain free, they can give up momentum provided the process by which this is done does not involve transference of energy.

Bound Energy per Unit Volume in the Electric Field.

The velocity at any place in a fluid containing vortex filaments is affected by filaments which are a considerable distance away, and we have to know the distribution of these to calculate the energy at the place. We are not, however, in the electrical problems concerned with the whole of the energy, but only with the part which is bound. A reference to fig. 2 will show that the energy bound by a filament is quite close to it, so that in calculating the contribution of a single filament to the energy in the field we need only consider the fluid within a short distance of the filament. If a is this distance, then the contribution of a filament to the bound energy will

be its contribution to the energy in a cylinder whose axis is the filament and whose radius is a. Let O be the centre of the normal section of this cylinder, P the place where another parallel filament crosses this plane. We know (Lamb's 'Hydrodynamics,' 5th edit. p. 205) that the motion of the liquid *inside* a closed cylinder due to a filament at P is that due to the filament itself plus that due to a filament of equal strength but opposite sign placed at the image Q of P in the cylinder, $OP.OQ=a^2$, where a is the radius of the cylinder. The energy per unit length of two parallel filaments of strength m_1, m_2 at a distance r_{12} apart is

$$-\frac{\rho}{\pi}m_1m_2\log r_{12},$$

where ρ is the density of the fluid; apply this to the filaments

$$+m \text{ at } 0, \quad +m \text{ at } P, \quad -m \text{ at } Q,$$

and we find that the energy equals

$$\frac{\rho}{\pi}m^2\log\frac{OQ}{OP}=\frac{\rho}{\pi}m^2\log\frac{a^2}{OP^2}.$$

If there are other filaments at P_2, P_3, P_n inside this cylinder the energy within it will be

$$\rho\frac{m^2}{\pi}\Sigma_1^n\log\frac{a^2}{OP_s^2},$$

or, if there are n filaments besides the one at O,

$$\rho\frac{m^2}{\pi}.n \text{ (mean value of } \log a^2/OP^2);$$

for a uniform and continuous distribution of filaments the mean value is unity, and we shall take this as an approximation. Thus the contribution of the filament we are considering to the energy is $\rho nm^2/\pi$. If σ is the number of filaments passing through unit area $n=\sigma\pi a^2$, and the contribution of the filament is $\rho\sigma m^2a^2$; as there are σ filaments passing through unit area the energy per unit volume is $\rho\sigma^2m^2a^2/2$. The factor one half is introduced, since each term, such as $m_1m_2\log r_{12}$, has been counted twice over, once when considering the cylinder whose axis is m_1 and again when considering

the one whose axis is m_2. The energy per unit volume is equal to $R^2/8\pi$, where R is the electric force ; hence

$$\frac{R^2}{8\pi} = \frac{\rho\sigma^2 m^2 a^2}{2}$$

or

$$R = 2.\sigma m.a\sqrt{\pi\rho}.$$

This agrees with our assumption that the force is proportional to the number of unit filaments passing through unit area. When the force at a charged surface is R the electricity on unit area is $R/4\pi$; this is the charge at the end of σ filaments, hence the charge at the end of each filament is

$$\tfrac{1}{2}ma\sqrt{\rho/\pi}.$$

Change of Mass due to the Motion of Electric Charges.

The energy per unit volume in the electric field is proportional to the number of granules gripped by the vortex filaments, and this, as we have seen, is proportional to the square of the electric force, *i. e.*, is proportional to the square of the number of filaments passing through unit volume. Now when the sum of a number of quantities is fixed the sum of their squares is a minimum when all the quantities are equal ; it follows from this that when the total number of filaments starting from a given area is fixed the number of granules bound by the filaments, and therefore the bound mass, will be least where the filaments are uniformly distributed over the area, *i. e.*, when the electric force over the area is uniform.

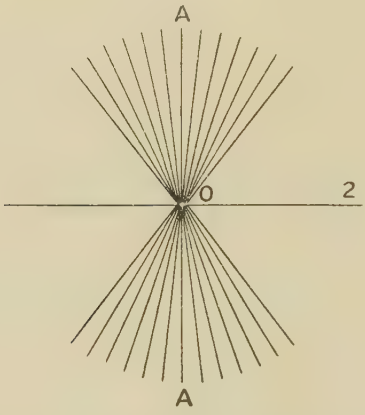
An electron when in motion carries its filaments along with it. The motion of a filament will produce the same motion in the fluid around it as would a long thin cylinder of the same shape and size. Now (see Lamb's ' Hydrodynamics,' p. 156) the motion of a long cylinder through a fluid gives rise to a couple on the cylinder, tending to make it set itself at right angles to its direction of motion, whose moment is equal to

$$\rho\pi a^2 l v^2 \sin\theta\cos\theta,$$

where ρ is the density of the fluid, a the radius of the cylinder, l its length, v the velocity, and θ the angle v makes with the axis of the cylinder. This couple will make filaments crowd towards the equatorial plane OA of the electron and leave the polar region around

OZ, the direction of motion of the electron. This will go on until the increase in the repulsion due to the crowding of the filaments in the equatorial region balances the effect due to the movements of the filaments. The distribution of the filaments will be no longer uniform, and hence their bound energy and therefore the mass will be greater than that when the electron is at rest. This on our view is the origin of the variation of electronic mass with velocity. The electronic filaments can bind more granules when they are in motion than when they are at rest, so that granules which were free when the electron was at rest are bound to it when it is in motion.

Fig. 3.



When the velocity diminishes the electron sheds some of the granules it had bound and these again become free. The sums of the bound and free masses and the bound and free energies are both constant. Thus the sum of the kinetic energy of a system and the energy of the free granules around it is constant.

The idea of bound and free mass and energy has an important bearing not only on the variation of mass with velocity but also on the nature of what is meant by " Potential Energy " in ordinary dynamics. In one sense neither the mass nor the energy of the free granules is mass or energy at all, since they do not form part of the mass or energy of *matter*, and so do not directly affect our senses; they have, however, the potentiality

of becoming mass or energy when they are gripped by the tubes of force attached to matter. So, too, the mass or energy of ordinary matter has the potentiality of becoming free mass or free energy.

To illustrate the nature of potential energy we shall take two concrete cases.

Consider the following very simple problem.

What is the difference between the energy bound to an electron held at rest (1) when it is not, and (2) when it is, exposed to a uniform electric force, when in the ordinary view it would have potential energy ? Consider the energy and what is proportional to it, the mass enclosed within a sphere of radius R, with its centre at the electron; let X be the external force supposed parallel to the axis of x, e the charge on the electron. The energy per unit volume is $F^2/8\pi$, where F is the electric force at the point x, y, z; if r is the distance of this point from the electron

$$F^2=\left(X+\frac{ex}{r^3}\right)^2+\frac{e^2y^2}{r^6}+\frac{e^2z^2}{r^6}$$

$$=X^2+2e\frac{Xx}{r^3}+\frac{e^2}{r^4}.$$

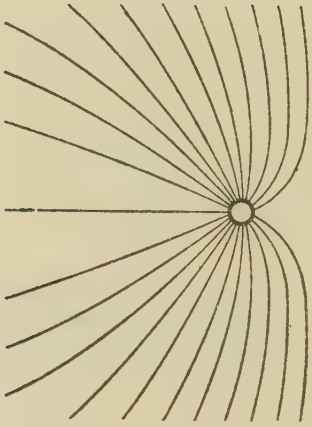
Integrating this expression through the space between the electron and the sphere, we find that the energy inside the sphere, since the second term in the expression for F^2 vanishes on integration, is equal to

$$\frac{X^2}{8\pi}\frac{4}{3}\pi\,(R^3-a^3)+\frac{e^2}{2a}-\frac{e^2}{2R},$$

a being the radius of the electron. The first term is the energy in the space between the electron and the sphere which would be bound to the external field if the electron was not present; the other terms are the energy in the same space which would be bound to the electron if there were no external electric field. The actual amount of bound mass when both electron and electric field are present is the sum of these two quantities, indicating that the amount of energy bound by either is not influenced by the presence of the other. The change which has been produced by the electric field is that the tubes of force starting from the electron, which in the absence of the field were uniformly distributed in all directions,

are concentrated under the action of the field on the left-hand side of the electron, where they point in the direction of the external electric force, at the expense of those on the right-hand side. Granules have moved from the right-hand side to the left of the electron, but the total number of bound granules has not changed. The tension in the tubes on the left will overpower the tension of those on the right, and the electron, if free, would begin to move in the direction of the electric force. In spite of its motion its energy would not increase until it had acquired a speed great enough to enable its tubes of force to grip one more granule than they could

Fig. 4.



when at rest; the electron will then acquire a granule, and with it the energy possessed by the granule when free; as the speed of the electron increases it takes up more and more granules and acquires more and more energy. This energy has been obtained at the expense of the energy of the free granules; granules which were free have been bound by lines of force to the electron, and the increase in the energy bound to the electron is equal to the energy lost by the free granules. Thus, if we regard potential energy as the kinetic energy of the free granules the conservation of energy is a truism.

Let us consider the case of the electron under electric force from another point of view. The effect of the force is to produce greater density of the tubes of force,

and therefore of bound energy and mass, on the left side of the electron than on the right. An increase in bound mass implies a decrease in free mass, so that the density, and consequently the pressure, due to the free granules is less on the left than on the right side of the electron. The difference in pressure will make the electron move from right to left, *i. e.*, in the direction it would move under the action of the electric force. The diminution in pressure will be equal to the increase in the bound energy per unit volume.

If p is the pressure when the electric force is R, Π that when R=0,

$$p=\Pi-\frac{\mathrm{R}^2}{8\pi}.$$

If the electric force at the surface of an electron is normal to its surface, R, the electric force at a point where the radius makes an angle θ with the axis of X is equal to that at a charged conducting sphere; hence

$$\mathrm{R}=-\frac{e}{a^2}+3\mathrm{X}\cos\theta.$$

The force tending to move the electron in the direction of the electric force is $\int p\cos\theta.d\mathrm{S}$ taken over the surface of the electron. When p has the value given above this integral is equal to Xe, which is the right expression for the force acting on the electron. Thus the force is that which would be produced by a distribution of pressure equal to the energy of the free granules per unit volume. Thus we may regard the potential energy per unit volume as represented by the pressure of the free granules.

If the electron A is not in a uniform field but in one due to a charge e' at B at a distance d from A, then the energy inside a sphere radius R with its centre at A which is not bound to the charge e' is

$$\frac{1}{2}\frac{e^2}{a}-\frac{1}{2}\frac{e^2}{\mathrm{R}} \quad \text{if } \mathrm{R}<d, \text{ and}$$

$$\frac{1}{2}\frac{e^2}{a}-\frac{1}{2}\frac{e^2}{\mathrm{R}}+ee'\left(\frac{1}{d}-\frac{1}{\mathrm{R}}\right);$$

if $\mathrm{R}>d$, a is the radius of the charge e at A, E the charge at B. Thus the mass of A will be affected by the position and the charge on B. It is evident that this

should be so, for some of the lines of force starting from A may end on B, and if the distance between A and B be increased the length of these will be increased and thereby the mass they can bind. This additional mass bound to A will be situated at a distance $\leqslant d$ from A, and similarly the additional energy bound by B will be in the neighbourhood of A rather than at B.

The sum of the energies bound by e and e', if b is the radius of the charge e', is

$$\frac{1}{2}\frac{e^2}{a}+\frac{1}{2}\frac{e'^2}{b}+\frac{ee'}{d}.$$

On the view that potential energy is the energy of the free granules and kinetic energy that of the bound the transformation of potential into kinetic energy is just the passage of a granule from one side to another of a surface like CC in fig. 2, p. 813. The expression transformation of energy is apt to convey a wrong impression—it is rather a transference than a transformation, for in each case the energy is the kinetic energy of the granules.

Maxwell's Equations.

We regard the motion of the granules as consisting of a drift or wind whose velocity components are $\bar{u}$, $\bar{v}$, $\bar{w}$, superposed upon a distribution of velocities where all directions of motion are equally probable, as in a gas in the normal state; if U, V, W are the components of these velocities, u, v, w those of the granule,

$$u=\bar{u}+\mathrm{U},$$
$$\bar{\mathrm{U}}=\bar{\mathrm{V}}=\bar{\mathrm{W}}=0,$$
$$\bar{\mathrm{U}}^2=\bar{\mathrm{V}}^2=\bar{\mathrm{W}}^2\equiv s^2\equiv 1/k.$$

The motion is in part rotational, the rotational part being represented by the presence of vortex filaments all of the same strength. Let the algebraic sum of the strength of the filament passing through unit area of a plane at right angles to x, y, z respectively be denoted by ξ, η, ζ, and let

$$\alpha=\bar{v}\zeta-\bar{w}\eta,$$
$$\beta=\bar{w}\xi-\bar{u}\zeta,$$
$$\gamma=\bar{u}\eta-\bar{v}\xi.$$

Then the following equations are satisfied (J. J. Thomson, Phil. Mag. xi. p. 1057, 1931):

$$4\pi\frac{d\xi}{dt}=\frac{d\gamma}{dy}-\frac{d\beta}{dz},$$

$$4\pi\frac{d\eta}{dt}=\frac{d\alpha}{dz}-\frac{d\gamma}{dx},$$

$$4\pi\frac{d\zeta}{dt}=\frac{d\beta}{dx}-\frac{d\alpha}{dy},$$

$$\frac{d\alpha}{dt}=\frac{d}{dz}\left(\frac{4\pi}{k}\eta\right)-\frac{d}{dy}\left(\frac{4\pi}{k}\zeta\right),$$

$$\frac{d\beta}{dt}=\frac{d}{dx}\left(\frac{4\pi}{k}\zeta\right)-\frac{d}{dz}\left(\frac{4\pi}{k}\xi\right),$$

$$\frac{d\gamma}{dt}=\frac{d}{dy}\left(\frac{4\pi}{k}\xi\right)-\frac{d}{dx}\left(\frac{4\pi}{k}\eta\right).$$

These are identical with Maxwell's equations if we identify (1) ξ, η, ζ with the components of Maxwell's electric displacement (this follows at once, since ξ, η, ζ are taken to be tubes of electric force); and (2) α, β, γ with the components of magnetic force. These expressions for the magnetic force are also those for the components of the acceleration acting per unit length of the filament due to its motion u, v, w (Lamb, 'Hydrodynamics,' p. 180). They represent the well-known forces acting on a rotating body when moving through a fluid.

Since Maxwell's equations are satisfied all the phenomena which follow from these equations can be illustrated by our model of the electric field, in particular the propagation of transverse waves of electric and magnetic force. We find that all the quantities satisfy the wave equation

$$\frac{d^2\phi}{dt^2}=\frac{1}{k}\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right).$$

Thus the velocity of the waves $=\sqrt{1/k}=s$, where

$$s^2=U^2=V^2=W^2$$

is the velocity of sound through the gas formed by the granules. As far as the motion of the granules goes these waves are longitudinal. But the motion of the granules is associated with the motion of the vortex

filaments and of the lines of magnetic force; each of these is at right angles to the velocity of the granules with which it is associated, so that a longitudinal wave of granules will correspond to a transverse wave of electric and magnetic force.

We see that Maxwell's equations when obtained in this way are equations between the average values of the quantities concerned; they are like the equations in the 'Kinetic Theory of Gases,' equations of statistical and not of ordinary dynamics.

Connexion between Velocity of Waves and Energy.

The velocity of waves through the assemblage of granules is, like that of sound, equal to the square root of

$$\tfrac{1}{3}(\overline{U}^2+\overline{V}^2+\overline{W}^2).$$

The relation between mass and energy is not, however, the same in the two cases. In a gas whose density is ρ the energy per unit volume is

$$\tfrac{1}{2}\rho(\overline{U}^2+\overline{V}^2+\overline{W}^2)=\tfrac{3}{2}\rho\ (\text{velocity of wave propagation})^2.$$

In a gas a given velocity of a molecule will contribute the same amount to the energy whatever its direction. It is not so with the granules, for here we have to consider not the whole energy but only the bound energy, *i. e.*, the energy bound to the vortex filaments. The granules bound by a filament are those moving at right angles to its length; an increase in the number of those moving parallel to it would not increase the bound energy. We see from the expressions for the magnetic force that motion of a filament along its length, *i. e.*, when $\xi/u=\eta/v=\zeta/w$, does not produce any magnetic force, and, again, we know from 'Hydrodynamics' (Lamb, p. 201) that the energy T due to a distribution of vortex filaments in a fluid whose density is ρ is given by the equation

$$T=2\rho.\int\int\int(x(w\eta-v\zeta)+y(u\zeta-w\xi)+z(v\zeta-u\eta)dx\,dy\,dz.$$

So that the component of the velocity of the granules along the direction of the filament at x, y, z does not contribute anything to the energy in the element $dx\,dy\,dz$.

If the directions of the velocities of the granules are uniformly distributed, then

$$\overline{U}^2=\overline{V}^2=\overline{W}^2=s^2,$$

whatever be the direction of the axes of coordinates. Take one of these, say z, parallel to the filament at x, y, z. Then since W^2 does not count, the bound energy per unit volume is

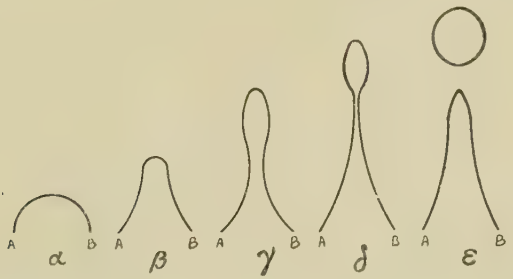
$$=\tfrac{1}{2}\rho(\bar{U}^2+\bar{V}^2)=\frac{\rho}{3}(\bar{U}^2+\bar{V}^2+\bar{W}^2)=\rho s^2,$$

where s is the velocity of propagation of the waves. This agrees with the ratio between energy, mass, and wave propagation, *i. e.*, $E=mc^2$, given by relativity.

The Photon.

Maxwell's equations, however, do not give any indication of the concentration of energy in units or quanta such as is implied by the existence of photons. We proceed

Fig. 5.



to consider whether the model of the electric field we are considering would suggest the existence of such bodies.

From the point of view we have taken the emission of radiation from an atom may be regarded somewhat as follows :—The energy in the radiation before it was emitted from the atom was bound to short tubes of force stretching between a proton and an electron ; the tube is anchored at each end, and the energy cannot get away from the atom unless part of the tube escapes from this anchorage and is free to move through space.

We may picture the process of detachment as somewhat of the kind shown in fig. 5 ; A and B are the photon and electron ; in α the tube is starting to escape from the atom, in β the process has got further, the left- and right-hand sides of the tube point in different directions, and therefore attract each other; the result of this attraction is showing itself in γ, and in δ the two sides of the

tube have come into contact; in ϵ fissure has taken place, the lower part of the tube is drawn back into the atom and the upper goes off as a closed ring, the photon.

The reverse of this process will illustrate the conversion of the energy of a photon into the energy of a tube in the atom connected with an electron, and therefore the communication of energy from a photon to an electron.

Thus from this point of view the existence of photons or quanta is essential for the emission of radiation from atoms or molecules.

In addition to the processes going on inside the atom, and which are little if at all affected by conditions external to it, there are, when an electric discharge is passing through a gas, a number of other electrical disturbances going on, such as occur when an atom is ionized or when a positive ion is neutralized by combination with an electron, or when an atom is thrown into an excited state. These would tend to make tubes of force spread outwards from their bases and to give off quanta; the energy in these would range within wide limits, since it would depend upon a variety of effects which occur with an almost random distribution of magnitude. The spectrum produced by these quanta would thus be continuous through a certain range of frequencies.

The photon is associated with a definite frequency ν, given by Planck's relation $E=h\nu$, where E is the energy of the photon. Since $E=mc^2$, $\nu=mc^2/h$, and thus is proportional to the mass of the photon, which on our view is proportional to the number of granules bound by it.

Since a photon does not lose energy when travelling through space the number of granules in it must keep constant; if it loses one at any time this must be immediately replaced by another from the sea of granules in which it is moving.

The granules in the photon will be struck by the free granules around it, and the number of impacts received by the bound granules in the photon per second from the free granules is, like the mass, proportional to the number of granules in the photon, so that if the impacts occurred at equal intervals Planck's Law would follow. But even if the impacts were not timed to follow each other at regular intervals, but occurred absolutely at random, so that the chance of an impact occurring in any

interval is proportional to the length of the interval, it can be shown (see J. J. Thomson, Proc. Manchester Phil. Soc. lxxv. p. 77, 1930) that when the number of impacts considered is very great the most probable distribution of the impacts is one where the great majority are separated by equal intervals, and the probability of this increases with the number of intervals. The impacts would thus produce an approximately regular disturbance, having a frequency proportional to the mass of the photon. On this view Planck's Law is a statistical one, and only holds when applied to a long sequence of vibrations.

The photon being a vortex ring will possess mass and energy due to the granules gripped by the vortex filaments which make up the ring. If we regard the ring as made up of circular vortex filaments all of the same strength, the energy and mass of the ring will depend upon the number of these filaments and upon their radii. When the ring is in a stable steady state the number of granules gripped will have a definite value. The filaments cannot grip more, and if one of the granules were detached from the ring another from the sea of granules through which it is moving would step in and replace it. We may regard the ring as analogous to a stable saturated compound of granules and vortex filaments immersed in an atmosphere of granules; the filaments are, as it were, saturated with granules and cannot take more, nor will the compound dissociate, *i. e.*, lose granules, since it is surrounded by multitudes of these. The constancy of mass and energy in the ring is thus assured by an automatic arrangement. The only way the energy or mass could be diminished would be by the Compton effect, *i. e.*, by a collision with an electron which detached from the photon one or more of the circular vortex filaments bound up in the vortex ring; the remainder would either go off as a photon of smaller energy than its parent or else give up its energy to the electron, as in the photoelectric effect.

A point of fundamental importance in relation to the properties of the photon is brought out by an experiment made by G. I. Taylor in the Cavendish Laboratory in 1909 (Proc. Camb. Phil. Soc. xv. p. 114, 1909), which proved that well defined interference patterns were produced by light having an intensity no greater than that

due to a standard candle at a distance of 1 mile, which corresponds to a flow of energy through 1 cm.^{2} of $1{\cdot}6\times10^{-6}$ ergs./sec. This involves that the number of photons per c.c. in the light was less than 2×10^{-5}, or less than one photon in 50 litres. This experiment proves that each photon must carry along with it all the materials for producing interference.

The Photon and Satellite Waves.

The photon, according to Planck's Law, has a frequency of vibration ν given by the equation $E=h\nu$. We have no means of measuring the frequency directly. It is determined indirectly by observing the interference and diffraction patterns produced by the light and determining the wave-length which would give rise to them. The question is, can we on the view we have taken of the electric field see any reason why the photon should be accompanied by a train of waves. The velocity of the photon, it must be remembered, is the same as that of waves through the free granules by which the photon is surrounded. Thus, if the photon were surrounded by an atmosphere of these waves they would travel in an invariable relation with it in its journey through space; the photon and its waves would thus travel as an invariable unit. We may picture a beam of light as consisting of isolated quanta moving through the free granules which are distributed through space and accompanied by a train of waves of wave-length c/ν travelling at the same speed as the photon. To produce interference fringes as sharp as those of light the train must contain several wave-lengths, and also its linear dimensions at right angles to the direction of propagation must be on the same scale, so that the dimensions of the unit consisting of the photon and its waves must be considerably greater than the wave-length of the light to which it corresponds. The dimensions of the photon itself are probably very small compared with those of the wave-train.

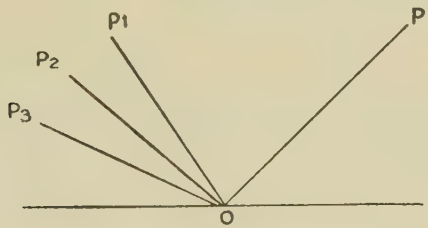
The waves accompanying the photon on this view are of the type of sound waves, but the interference effects produced by them will be of the same character as those produced by light waves of the same wave-length. In particular they will be diffracted by a grating in the same way. Thus, if, for example, PO represents

a beam of these waves incident upon a grating, it will by diffraction be concentrated into beams OP_1, OP_2, OP_3 &c., and $OP_{\bar{1}}$, $OP_{\bar{2}}$ . . . will have the same directions as the bright lines in the diffraction spectrum of light of the same wave-length.

That longitudinal waves such as sound waves exert pressures analogous to the pressure of light, and are accompanied by a stream of momentum in the direction in which the waves are travelling, has been shown by the mathematical investigations of Rayleigh *, Poynting †, Havelock ‡, Weaver §, and has been detected experimentally by R. W. Wood ||, Altberg ¶, and by experiments on the heroic scale by R. W. Wood and Loomis **.

We may suppose that in consequence of the longitudinal waves in the ocean of free granules through which the

Fig. 6.



photon is moving there is in this fluid a velocity which on the average is in the direction of propagation of the waves, independent of that arising from the vorticity in the vortex ring which we suppose forms the photon. We now proceed to investigate the influence of this independent velocity on the motion of the vortex ring.

The problem we have to solve is that illustrated in fig. 6. The vortex ring before reaching the grating is moving in the direction PO, and this is also the direction of momentum in the waves around it. By diffraction the waves will be diverted at the grating into the directions OP_1, OP_2 . . ., and their momenta will be in these directions. Thus, when the vortex ring reaches the grating it will

* Rayleigh, Phil. Mag. x. p. 364 (1905).
† Poynting, 'Collected Papers,' p. 335; Phil. Mag. ix. p. 393 (1905).
‡ Havelock, Phil. Mag. xlvii. p. 754 (1924).
§ Weaver, Phys. Review, xv. p. 399 (1920).
|| R. W. Wood, *Physik. Zeits.* vi. p. 22 (1905).
¶ Altberg, *Ann. der Physik*, xi. p. 405 (1903).
** R. W. Wood and Loomis, Phil. Mag. iv. p. 417 (1927).

get into a stream where the flow is not in the direction in which it is moving; we have to find whether the ring will fight its way across the stream or be turned so as to move along it.

The components of momentum P, Q, R parallel to the axes of x, y, z respectively, due to a vortex ring of vorticity m moving in a fluid of density ρ, are given by equations of the type

$$\mathrm{P}=m\rho\int\left(y\frac{dz}{ds}-z\frac{dy}{ds}\right)ds=2m\rho.\mathrm{A}\lambda,$$

where A is the area of the aperture of the ring, λ, μ, ν the direction cosines of the plane of the aperture; ds is an element of the circumference of the ring.

T, the energy due to the ring, is given by the equation

$$\mathrm{T}=2m\rho.\int\left\{u\left(y\frac{dz}{ds}-z\frac{dy}{ds}\right)+v\left(z\frac{dx}{ds}-x\frac{dz}{ds}\right)+w\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)\right\}ds,$$

where u, v, w are the components of the velocity of ds. If u, v, w are the same at all points of the ring, this gives

$$\mathrm{T}=2(\mathrm{P}u+\mathrm{Q}v+\mathrm{R}w)=4m\rho\mathrm{A}(\lambda u+\mu v+\nu w).$$

Take the axis of z in the direction of propagation of the waves through the granules; then w is the sum of two parts, one, w_0, due to the vorticity, the other, W, to the stream produced by the waves. The part of the energy which concerns our problem is that depending on W, *i. e.*,

$$4m\rho\mathrm{A}\cos\theta\mathrm{W},$$

where θ is the angle between the normal to the aperture of the ring and the direction of the stream due to the waves. Since this term occurs in the expression for the kinetic energy, there will be a couple tending to increase θ, whose moment is equal to the differential coefficient of this term with respect to θ; thus the moment is

$$-4m\rho.\mathrm{A}\sin\theta.\mathrm{W}.$$

We might have anticipated this result, for this expression represents the couple acting on a ring of area A carrying an electric current $4m\rho$ when placed in a uniform magnetic field of strength W making an angle θ with the normal to the plane of the ring.

There is a close connexion between the magnetic force produced by electric currents and the velocities produced by vortex filaments. The velocity at a point P due to a vortex filament has the same direction and is proportional in magnitude to the magnetic force at P which would be produced if the vortex filament were replaced by an electric current, the strength of the current being proportional to the strength of the filament. Again, the energy in the fluid is equal to

$$\tfrac{1}{2}\int\int\int\rho(u^2+v^2+w^2)\,dx\,dy\,dz,$$

where u, v, w are the components of the velocity at x, y, z.

The energy in the electric field is

$$\tfrac{1}{8}\pi\int\int\int(\alpha^2+\beta^2+\gamma^2)dx\,dy\,dz,$$

where α, β, γ are the components of the magnetic force. Thus, if we know the expression for the energy in the electric problem we can get that for the vortex ring one by replacing the currents by vortex filaments and magnetic force by velocity; and since in the electric problem there is a couple tending to make the plane of the electric circuit place itself at right angles to the external magnetic force, there will in the vortex problem be a couple tending to make the ring place itself at right angles to the current in which it is swimming, and so that the velocity of the ring due to its vorticity is in the same direction as that due to the current.

We shall now endeavour to form an estimate of the distance the vortex ring will travel before it is swimming with the current. Let M be the bound mass of the ring, Mk^2 its moment of inertia about an axis in the plane of the ring. Thus, using the preceding expression for the couple, we get

$$Mk^2\frac{d^2\theta}{dt^2}+4\pi\rho ma^2W\sin\theta=0,$$

where a is the radius of the aperture of the vortex ring. This equation is of the same type as that for the motion of a pendulum, and corresponds to a vibration about $\theta=0$, whose frequency n is given by

$$\frac{n}{2\pi}=\cdot\left\{\frac{4\pi\rho ma^2W}{Mk^2}\right\}^{1/2}.$$

θ will be given, if $\dot{\theta}=0$ when $t=0$, by the equation

$$\theta=\theta_0 \cos\frac{nt}{2\pi}.$$

The velocity of the ring parallel to the stream is $c\cos\theta$ if c is the velocity of the ring, *i. e.*,

$$c\cos\left(\theta_0\cos\frac{nt}{2\pi}\right);$$

this is always positive, and the average velocity is greater than $c\cos\theta_0$.

The velocity at right angles to the stream is

$$c\sin\left(\theta_0\cos\frac{nt}{2\pi}\right),$$

and the average velocity is zero, so that the vortex ring zig-zags backwards and forwards across the stream. The ring will only travel in one direction for a time π/n, and the distance it travels in that direction will be less than $\pi c/n$.

We can estimate the order of n as follows. Since we are only considering orders of magnitude, we may put $k^2=a^2$. One expression for the momentum of the ring is Mc, another is $2m\rho\dot{\mathrm{A}}=2m\rho\pi a^2$; evaluating M by this relation we get

$$\frac{n}{2\pi}=\left(\frac{2\mathrm{W}c}{a^2}\right)^{1/2}, \text{ or if } \mathrm{W}=c/\mathrm{N},$$

$$\frac{n}{2\pi}=\left(\frac{2}{\mathrm{N}}\right)^{1/2}\frac{c}{a};$$

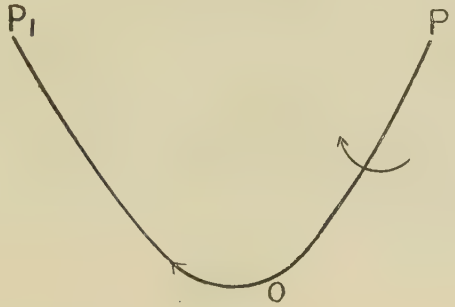
inserting this value of n in the expression $\pi c/n$ for the maximum distance travelled across stream, we find that this is equal to

$$a\sqrt{\mathrm{N}}/2\sqrt{2}.$$

Thus, if the velocity W were only one thousandth part of the velocity of the ring the greatest distance the ring could travel across stream would be about twelve times the radius of the photon, which in all probability is a very small fraction of the wave-length of the light with which it is associated. Thus, if a photon once gets into one of the streams OP_1, OP_2, OP_3, it will never get out of it, and the path of the proton will be guided by the waves in the medium through which it is moving.

We have yet to prove that, assuming all the energy is in the photon, the relative amounts of energy in the spectra of different orders are the same as that given by the Undulatory Theory. To do this start from a point in the spectrum of the nth order, and travel backwards along the direction of the Poynting vector, *i. e.*, along the direction in which the energy travels, until we arrive at a point on the plane front of the incident wave before it reached the grating. If we do this for all points in this spectrum we shall find that its energy came from a series of patches on the wave front. If the sum of the areas of these patches is A_n, then on the Undulatory Theory the energy in the nth order spectrum is proportional to A_n, and the ratio of the energies in the

Fig. 7.



nth and mth order spectra is A_n/A_m. If the photon is in the area A_n it will be carried to the nth spectrum, if in A_m to the mth. If the photon was distributed at random across the wave front then the ratio of the probability of its going to the nth or mth spectrum is A_n/A_m, *i. e.*, the same as the ratio of intensities on the Wave Theory.

The circular vortex ring has no vector property other than that due to the orientation of the plane of its circular aperture, and this is determined by the direction in which the ring is moving. Thus two vortex rings of the same strength and radius, and moving in the same direction, are alike in all respects. The longitudinal waves through the free granules have also no vector property except that associated with the direction of propagation. Thus the combination of vortex rings and longitudinal waves could not give rise to polarization. If, however, the vortex

rings had a spin about axes through their centres and at right angles to the plane of their aperture these vortex rings, moving in the same direction, might be differentiated from each other by the direction of their spin.

For example, when the vortex ring is in the transitional state between moving in its original direction PO and its final direction OP_1 it will, when the spin is in the direction indicated in fig. 7, be acted on by a force at right angles to the plane of the paper and downwards, while if the spin were in the opposite direction the force would be upwards. Thus it might be possible to separate those spinning in one direction from those spinning in the opposite, and thus get plane polarized light.

On a Superior Limit to the Mass of a Granule.

On the view we have taken the difference between the mass of any two systems cannot be less than the mass of a granule.

The masses of which we have accurate measurements which differ least from each other are probably those of photons corresponding to spectroscopic lines of nearly equal wave-length.

If ν is the frequency of a line, then the mass M of its photon is given by

$$\nu = \frac{Mc^2}{h},$$

or

$$\delta\nu = \delta M . \frac{c^2}{h}.$$

It is more convenient to express δM in terms of the wave-length λ then, since

$$\frac{\delta\nu}{\nu} = -\frac{\delta\lambda}{\lambda},$$

$$\delta\nu = -\delta\lambda . \frac{\nu}{\lambda} = -\frac{\delta\lambda . c}{\lambda^2}.$$

Thus

$$\frac{c\delta\lambda}{\lambda^2} = \delta M . \frac{c^2}{h} \quad \text{or} \quad \frac{\delta\lambda}{\lambda^2} = \frac{\delta M \times 3 \times 10^{10}}{6{\cdot}5 \times 10^{-27}}.$$

Suppose we can separate by spectroscopy two lines where

$$\delta\lambda/\lambda = 10^{-6} \quad \text{and} \quad \lambda = 3 \times 10^{-5} \text{ cm.};$$

then we have

$$\frac{\delta\lambda}{\lambda^2} = \frac{1}{30},$$

and hence

$$\delta M = \cdot 7 \times 10^{-38} \text{ gm.}$$

On these hypotheses the mass of a granule must be less than $1/10^{10}$ of that of an electron.

The bound mass of matter is obtained at the expense of the free granules in its neighbourhood; for a proton assumed to have a radius of 10^{-13} cm. the density of the matter is of the order 10^{14} (gr./cm.^{3}). The absorption of free granules required to produce such a density as this might well lower the density of the free granules in the neighbourhood. This would diminish the bound mass of an electron in its neighbourhood and the tension in the vortex filament, and this would be accompanied by a change in the law of force between the electron and the proton.

The normal pressure, p, of the granules is reduced to $p - R^2/8\pi$ at a place where the electric force is R; at the surface of the cavity from which the lines of force start the pressure vanishes, hence if a is the radius of the cavity

$$p = \frac{1}{8\pi}\frac{e^2}{a^4}$$

or

$$a^4 = \frac{e^2}{8\pi p}.$$

Thus, since all the electrons have the same charge they must all be of the same size; it would follow too that, since the charge on the proton is the same as that on an electron, the radius of the proton, and therefore the energy and mass, might be the same as that of an electron. The recent discovery of positive electrons is in accordance with this view, but in general the positive charge is associated with a radius much smaller than that of an electron.

The energy, E, when the radius is r, if r is great enough to make the law of the inverse square apply, is given by

$$E = \frac{4}{3}\pi p r^3 + \frac{e^2}{2r}.$$

The graph when E is ordinate and r abscissa is represented by fig. 8. For stable equilibrium E must be a minimum, and therefore

$$r^4 = \frac{e^2}{8\pi p},$$

the value obtained before.

Fig. 8.

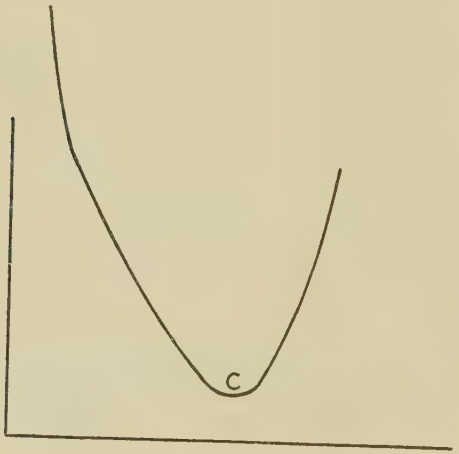
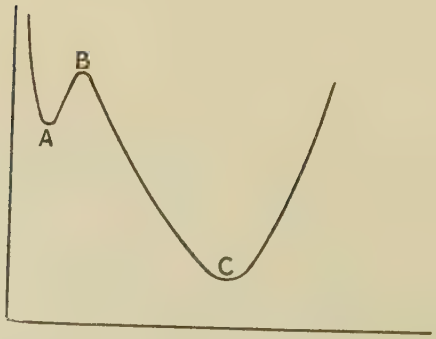


Fig. 9.



We have, however (p. 835), given reasons for supposing that when r is comparable with the size of the nucleus the law of force may be other than the inverse square, so that for small values of r the graph for E and r might be expected to be different from fig. 8. If it were like that in fig. 9 E would have a minimum at A, a maximum at B,

and another minimum at C, the graph before reaching C becoming the same as that of the corresponding part of fig. 8, which holds for the larger values of r. Here a radius equal to the abscissa of A corresponds to a stable state as well as that of C; so that the same charge might be carried by cores of two different sizes, one much larger than the other. If the energy and radius of the proton were represented by the point A on the curve, then if it were to receive an amount of energy greater than that represented by the difference of the ordinates at A and B it might be carried past the unstable position B, rapidly lose energy, and finally settle down to the position represented by C, emitting while doing so energy represented by the difference between the ordinates of B and C. It would emit more energy than was required to carry it from A to B by the amount represented by the difference between the ordinates of A and C. In this way a large amount of the atomic energy it possessed when at A might be liberated. The charge would start as a proton at A and end at C as a positive electron, and give out in the process energy equal to the difference between the energy of a proton and that of an electron.

If an electron were to receive an amount of energy greater than that represented by the difference between the ordinates at C and B it might get past the stage B and settle down as a negative proton at A. The energy which must be given to the electron for this to happen is greater than the difference between the energy of a proton and that of an electron.

Isotopes and Neutrons.

A graph similar to that in fig. 9 would represent the relation between the radius and energy if the force at a distance r were equal to

$$\frac{e^2}{r^2}\left(1-\frac{\alpha}{r}+\frac{\beta^2}{r^2}\right),$$

where α and β are small quantities comparable with the size of the nucleus.

The change in the law of force at nuclear distances makes it possible for protons and electrons in the nucleus to combine in proportions which would not be possible at atomic distances. Thus, to take a simple case, if β were

small compared with α, so that the force at not too small distances is approximately

$$\frac{e^2}{r^2}\left(1-\frac{\alpha}{r}\right),$$

two protons separated by a distance $7\alpha/3$ with an electron midway between them would form a system in stable equilibrium. Since α is comparable with nuclear dimensions the system at the much greater atomic distances would act like a unit positive charge with a mass twice that of a proton, *i. e.*, like an isotope of hydrogen with an atomic weight 2. A proton and an electron with the same law of force would be in equilibrium when at rest if the distance between them were α, which is very small compared with atomic distances; such a system would have many of the properties of a neutron. This view suggests the possibility of the existence of neutrons with masses larger than that of a proton. For example, a system with two protons at the ends of one diagonal of a square and two electrons at the opposite ends of the other would, if the side of the square equals $1{\cdot}2\alpha$, be stable, and would form a neutron with atomic weight 2. One with atomic weight 4 might be formed if four protons and four electrons were arranged at the corners of a cube in such a way that each proton had three electrons for its nearest neighbours and each electron three protons.

Waves due to the Motion of Electrons and Protons.

The tubes of force which start from an electron or a proton are uniformly distributed over its surface when the electron is in equilibrium, and if a tube is displaced from its equilibrium position it will oscillate about it. If a tube is displaced through a distance ξ it will be acted upon by a force equal to $p^2\xi$ per unit mass, where $p=2\pi n$, n being the frequency of the vibrations.

Consider the behaviour of a tube of force which in equilibrium lies along the axis of x. Let P and Q (fig. 10) be two points on the displaced tube, $\triangle x$ the distance between them, y the vertical displacement of the tube at P, σ the mass of unit length of the tube, ρ the radius of curvature of its axis at P, and T the tension in the tube.

The equation of motion is

$$\sigma\triangle x\frac{d^2y}{dt^2}=\frac{\mathrm{T}}{\rho}\triangle x-p^2y\sigma\triangle x.$$

We have seen (p. 812) that T is equal to the energy per unit length of the tube, and that this energy

$$=mc^2=\sigma c^2.$$

Hence this equation becomes

$$\frac{d^2y}{dt^2}=\frac{c^2}{\rho}-p^2y.$$

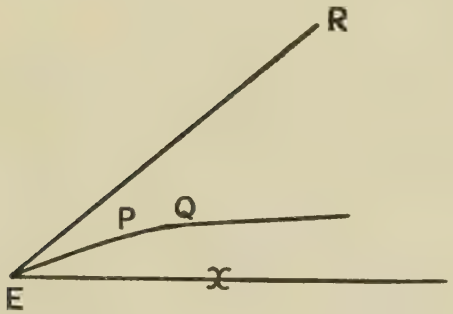
Neglecting squares of small quantities

$$1/\rho=d^2y/dx^2\ ;$$

hence

$$\frac{d^2y}{dt^2}=c^2\frac{d^2y}{dx^2}-p^2y, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Fig. 10.



so that if

$$y=\cos n\left(t-\frac{x}{\mathrm{V}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{c}{\mathrm{V}}=\frac{\sqrt{n^2-p^2}}{n}\equiv f(n), \quad . \quad . \quad . \quad . \quad (3)$$

and

$$y=\cos n\left(t-\frac{x}{c}f(n)\right).$$

The phase of this expression will be stationary at a place x and time t if

$$\frac{d}{dn}\left\{n\left(t-\frac{x}{c}f(n)\right)\right\}=0,$$

or

$$t-\frac{x}{c}(f(n)+nf'(n))=0,$$

or by equation (3)

$$t-\frac{x}{c}\frac{\mathrm{V}}{c}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When this condition is satisfied disturbances of slightly different frequencies will be in the same phase and will combine to produce an abnormally large disturbance, such as a hump in the tube of force. When the motion of the electron is steady this must be at rest relative to the electron, so that if the electron is moving along the axis of x with velocity u equation (4) must be satisfied by $x=ut$, and therefore $u\mathrm{V}$ must equal c^2.

V is the phase velocity and $n/\mathrm{V}=2\pi/\lambda$, where λ is the wave-length ; hence

$$\lambda u n=2\pi c^2,$$

or λ varies inversely as the velocity of the electron, a well-known property of electronic waves. We have also from (2)

$$y=\cos n\left(t-\frac{ux}{c^2}\right),$$

which agrees with the equation which would be given by the Lorentzian transformation.

Let us now consider another tube of force OQ, making an angle θ with the direction of motion of the electron. The differential equations for the propagation of a disturbance along this tube will be the same as for the previous one if s the distance measured along OQ be substituted for x.

The tube of force OQ is carried along by the electron as if rigidly connected with it. Hence the velocity of a point in the tube along OQ is $u\cos\theta$, and the equation

$$t-\frac{s}{c}\frac{\mathrm{V}}{c}=0$$

must be satisfied by $s=u\cos\theta.t$; this gives

$$u\mathrm{V}\cos\theta=c^2,$$

so that the phase velocity along OQ is $c^2/u\cos\theta$.

The disturbance along OQ is thus

$$\cos n\left(t-\frac{s}{\mathrm{V}}\right)=\cos n\left(t-\frac{us\cos\theta}{c^2}\right)=\cos n\left(t-\frac{ux}{c^2}\right).$$

Thus the disturbance in all the tubes coming from the electron are at the same time in the same phase when x is constant, and constitute a series of plane waves travelling parallel to the axis of x and having a wave-length equal to c^2/nu.

We notice too that

$$\frac{n^2-p^2}{n^2} = \frac{u^2}{c^2},$$

or

$$\frac{n}{p} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}},$$

a result which also agrees with that given by relativity.

The preceding expressions relate to disturbances propagated along the tubes of force, *i. e.*, vortex filament; these disturbances will produce periodic disturbances in the fluid through which the filaments are distributed; these will be propagated as waves, producing, as in the case of the photon, a flow of momentum in the direction in which the wave is travelling. When the direction of the wave changes, as in diffraction, the electron will be exposed to the action of the flow in the direction of the diffracted wave, and will by an effect analogous to that of the pressure of light be carried along in that direction.

Effects produced by the Passage of Granules through Matter.

Consider a cube of the matter bounded by planes at right angles to the axes of x, y, z respectively. Suppose the granules divided into six equal streams, two being parallel to each axis, one flowing in the positive the other in the negative direction.

Take, first, the stream in the positive direction of x:

let $f_1(\theta)\,d\theta/k_1$, where $k_1 = \int_0^\pi f_1(\theta)\,d\theta$,

be the probability that after collision with a molecule inside the cube the direction of motion of the granule makes an angle between θ and $\theta+d\theta$ with the axis of x. The velocity of the granule parallel to x is $c\cos\theta$, and the momentum parallel to x given to the molecule is

$$mc(1-\cos\theta).$$

The expectation of this is

$$mc\left(1-\frac{1}{k_1}\int_0^\pi f_1(\theta)\cos\theta\,d\theta\right).$$

The granules moving parallel to y or z will not by their collisions on the average change the momentum parallel to x.

Consider the deflected granule after it has passed out of the cube; its momentum parallel to x is $mc \cos \theta$, but as the granule is moving in a direction making an angle θ with the axis of x, the amount of this momentum carried across unit area at right angles to this axis in unit time is

$$mc \cos \theta \times c \cos \theta \quad \text{or} \quad mc^2 \cos^2 \theta,$$

and the expectation of this is

$$\frac{mc^2}{k_1} \int_0^{\frac{\pi}{2}} f_1(\theta) \cos^2 \theta \, d\theta.$$

The stream of granules moving in the negative direction of x will contribute to the stream of momentum coming out of the cube in the positive direction if the deflexion of the granule at a collision is greater than $\pi/2$; their contribution to the rate of flow through unit area at right angles to the x plane is

$$\frac{mc}{k_1} \int_{\frac{\pi}{2}}^{\pi} f_1(\theta) \cos^2 \theta \, d\theta,$$

so that the flow in the positive direction of x, due to the collision with a molecule of matter, of one granule moving in the positive direction and another moving in the negative, is

$$\frac{mc}{k_1} \int_0^{\pi} \cos^2 \theta . f_1(\theta) \, d\theta.$$

The granules moving parallel to y will, after collision, contribute to the flow of momentum in the positive direction of x. Let θ now be the angle the granule after collision makes with the axis of y, and ϕ the angle the plane through this axis and the deflected granule makes with the plane of xz; the cosine of the angle the deflected granule makes with the axis of x is $\sin \theta \cos \phi$. The momentum parallel to x is $mc \sin \theta \cos \phi$, and the rate it carries momentum through unit area of a plane at right angles to x, is

$$mc \sin \theta \cos \phi \times \sin \theta \cos \phi, \; i.\,e., \; mc^2 \sin^2 \theta \cos^2 \phi.$$

The chance that ϕ is between ϕ and $\phi+d\phi$ is $d\phi/2\pi$, and the chance that θ is between θ and $\theta+d\theta$ is $f_2(\theta)d\theta/k_2$, where $k_2=\int_0^\pi f_2(\theta)\,d\theta$. If the cube is crystalline $f_1(\theta)$ is not necessarily equal to $f_2(\theta)$. The expectation of the rate of flow in the direction of x is

$$\frac{mc^2}{2\pi k_2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\int_0^\pi \cos^2\phi\,\sin^2\theta\,f_2(\theta)\,d\phi\,d\theta$$
$$=\frac{mc^2}{4k_2}\int_0^\pi \sin^2\theta\,.\,f_2(\theta)\,d\theta.$$

This is for a granule moving in the positive direction of y; one moving in the opposite direction will make an equal contribution, so that the momentum due to one collison in each of the y streams will be

$$\frac{mc^2}{2k_2}\int_0^\pi f_2(\theta)\,\sin^2\theta\,.d\theta.$$

Similarly the momentum parallel to x, due to one collision in each of the z streams, is

$$\frac{mc^2}{2k_3}\int_0^\pi f_3(\theta)\,\sin^2\theta\,.d\theta\,;$$

adding these three expressions we find that the rate of flow of momentum parallel to the axis of x, due to one collision from each of the six streams, is

$$\frac{mc^2}{2}\int_0^\pi\left(2\cos^2\theta\,.\frac{f_1(\theta)}{k_1}+\sin^2\theta\left\{\frac{f_2(\theta)}{k_2}+\frac{f_3(\theta)}{k_3}\right\}\right)d\theta,$$
$$=mc^2\left(1-\int_0^\pi\left\{\frac{f_1(\theta)}{k_1}-\tfrac{1}{2}\left(\frac{f_2(\theta)}{k_2}+\frac{f_3(\theta)}{k_3}\right)\right\}\sin^2\theta\,d\theta\right)$$
$$=mc^2\left(1-\left(\alpha-\frac{\beta+\gamma}{2}\right)\right),$$

if
$$\alpha=\int_0^\pi\frac{f_1(\theta)}{k_1}\sin^2\theta\,.d\theta,$$

and β and γ are symmetrical expressions.

Similarly the flows of momentum parallel to y and z are respectively

$$mc^2\left(1-\left(\beta-\frac{\alpha+\gamma}{2}\right)\right),$$

$$mc^2\left(1-\left(\gamma-\frac{\alpha+\beta}{2}\right)\right).$$

If the cube is isotropic $\alpha=\beta=\gamma$, and the flow of momentum in each case is mc^2, which is the value when there is no cube. Thus the insertion of isotropic matter in the path of the granules produces no effect upon the flow of momentum outside. There is, for example, no shielding effect of the kind contemplated in Le Sage's theory of gravitation, though the granules resemble in many respects his ultramundane corpuscle.

When the matter is crystalline α, β, γ may be different, but whatever their values the sum of the flows parallel to x, y, z is $3mc^2$, the same as if no matter was present, and the flow outward through a sphere surrounding the cube is also not affected by the matter.

Suppose the cube is made of a uniaxial crystal, x being the axis of symmetry, so that $\beta=\gamma$:

flow of momentum parallel to $x=mc^2(1-(\alpha-\beta))$;
flow of momentum parallel to either y or $z=mc^2(1+\frac{1}{2}(\alpha-\beta))$.

When $\alpha>\beta$ the flow of momentum parallel to x outwards from the cube will be diminished, while the flow inwards will not be affected. Thus, outside the crystal near the x-face the inward flow exceeds the outward one and there will be a drift of granules towards the face which would tend to carry any body in the stream towards it; thus the x-faces would appear to *attract* bodies in their neighbourhood. On the other hand, the outward flow at the y and z faces will be greater than the inward, and these faces would tend to *repel* bodies in their neighbourhood.

We have (see p. 827) identified the number of collisions between the granules and the matter in a mass M with the frequency which by Planck's Law is associated with the mass; on this view the number of collisions is Mc^2/h. The expression just given for the flow of momentum in the different directions refer to one collision in each of the six streams; to get the total effect we must multiply each of them by $Mc^2/6h$, where M is the mass of the cube.

The drift of granules produced in the ether outside the crystal will affect the velocity of light passing through the drift, the velocity being increased by the velocity of drift in the direction of propagation of light. Thus a ray of light would be refracted on passing into the drift. The effect is of the same kind as that of wind on the velocity of sound, and the law of refraction as given

by Lord Rayleigh's 'Theory of Sound,' 2nd edition, vol. ii. p. 133.

The lack of symmetry in the distribution of the direction of motion of the granules would be affected by the collisions between the granules, and unless the collisions were of a special type would not extend beyond a distance from the crystal comparable with the free path of a granule. If, however, the collisions between two granules were like those between equal elastic spheres moving in one plane with the same velocity, when, after a collision between A and B, A travels along what would have been B's path if there had been no collision and B along A's, then, since the properties of A and B are identical, no change is produced when A and B are regarded as an aggregate, though there is if they are regarded as individuals. Thus any lack of symmetry of the distribution of the directions in which the granules are moving would not be affected by such collisions.

LXXII. *The Calculation of Modulation Products.* By A. C. Bartlett *. (Communication from the Research Staff of the M.O. Valve Co., Ltd., at Wembley.)

IF to the grid of a three-electrode valve voltages of two different frequencies, f_1 and f_2, are applied simultaneously, then, owing to the non-linear characteristic curve, the anode current will contain components of all frequencies of the form $mf_1 \pm nf_2$, where m and n are any integers. Such components are usually termed modulation products.

The characteristic of the valve may be taken as a power series

$$i = \mathrm{F}(v) = a_0 + a_1 v + a_2 v^2 + \ldots ,$$

where i is the anode current and v the grid voltage. If a voltage $b + v_1 \cos \omega_1 t + v_2 \cos \omega_2 t$ is applied to the grid, then by Taylor's Theorem

$$i = \mathrm{F}(b) + \sum_{n=1}^{\infty} \frac{1}{\lfloor n} (v_1 \cos \omega_1 t + v_2 \cos \omega_2 t)^n \frac{d^n}{db^n} \mathrm{F}(b).$$

From this series any modulation product may be picked out after all the powers of cosines have been expressed as

* Communicated by C. C. Paterson, M.I.E.E.

sums of cosines of multiple angles. Thus, if the component of frequency $\frac{\omega_1-\omega_2}{2\pi}$ is required, it is necessary to find the coefficient of $\cos\omega_1 t\cos\omega_2 t$ in the above expansion, which is a laborious process.

If the symbolic form of Taylor's Theorem

$$f(x+h)=e^{h\frac{d}{dx}}f(x)$$

is used, then, writing θ_1 and θ_2 for $\omega_1 t$ and $\omega_2 t$,

$$\begin{aligned} i&=\mathrm{F}(b+v_1\cos\theta_1+v_2\cos\theta_2)\\ &=e^{(v_1\cos\theta_1+v_2\cos\theta_2)\frac{d}{db}}\mathrm{F}(b)\\ &=e^{v_1\cos\theta_1\frac{d}{db}}e^{v_2\cos\theta^2\frac{d}{db}}\mathrm{F}(b). \end{aligned}$$

The Sonine expansion

$$e^{z\cos\theta}=\mathrm{I}_0(z)+2\sum_{n=1}^{\infty}\mathrm{I}_n(z)\cos n\theta,$$

where $\mathrm{I}_n(z)$ is the modified Bessel coefficient

$$\mathrm{I}_n(z)=\sum_{k=1}^{\infty}\frac{1}{k\lfloor n+k}\left(\frac{z}{2}\right)^{n+2k},$$

can now be applied to transform the operator $e^{v_1\cos\theta_1\frac{d}{db}}$, which contains powers of $\cos\theta_1$, into the operator

$$\mathrm{I}_0\left(v_1\frac{d}{db}\right)+2\sum_{n=1}^{\infty}\cos n\theta_1\mathrm{I}_n\left(v_1\frac{d}{db}\right),$$

which contains cosines of multiples of θ_1 to power unity only. Thus

$$\begin{aligned} i&=\mathrm{F}(b+v_1\cos\theta_1+v_2\cos\theta_2)\\ &=\left\{\mathrm{I}_0\left(v_1\frac{d}{db}\right)+2\sum_{n=1}^{\infty}\cos n\theta_1\mathrm{I}_n\left(v_1\frac{d}{db}\right)\right\}\\ &\quad\times\left\{\mathrm{I}_0\left(v_2\frac{d}{db}\right)+2\sum_{n=1}^{\infty}\cos n\theta_2\mathrm{I}_n\left(v_2\frac{d}{db}\right)\right\}\mathrm{F}(b). \end{aligned}$$

From this the coefficient of any term $\cos n_1\theta_1\cos n_2\theta_2$ is seen to be

$$4\mathrm{I}_{n_1}\left(v_1\frac{d}{db}\right)\mathrm{I}_{n_2}\left(v_2\frac{d}{db}\right)\mathrm{F}(b).$$

The method may be readily extended to the case where the anode current is controlled by a number of electrodes by using Taylor's Theorem for a number of variables in the form

$$f(x_1+h_1,\ x_2+h_2, \ldots)=e^{h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\cdots} f(x_1,\ x_2, \ldots).$$

For example, if to two electrodes of a valve, voltages $a+v_1\cos\theta_1$ and $b+v_2\cos\theta_2$ are applied,

$$\begin{aligned} i&=\mathrm{F}(a+v_1\cos\theta_1,\ b+v_2\cos\theta_2)\\ &=e^{v_1\cos\theta_1\frac{\partial}{\partial a}}\ e^{v_2\cos\theta_2\frac{\partial}{\partial b}}\mathrm{F}(a,\ b)\\ &=\left\{\mathrm{I}_0\left(v_1\frac{\partial}{\partial a}\right)+2\sum_{n=1}^{\infty}\cos n\theta_1\mathrm{I}_n\left(v_1\frac{\partial}{\partial a}\right)\right\}\\ &\times\left\{\mathrm{I}_0\left(v_2\frac{\partial}{\partial b}\right)+2\sum_{n=1}^{\infty}\cos n\theta_2\mathrm{I}_n\left(v_2\frac{\partial}{\partial b}\right)\right\}\mathrm{F}(a,\ b), \end{aligned}$$

and the coefficient of $\cos\theta_1\cos\theta_2$ is

$$\begin{aligned} &4\mathrm{I}_1\left(v_1\frac{\partial}{\partial a}\right)\mathrm{I}_1\left(v_2\frac{\partial}{\partial b}\right)\mathrm{F}(a,\ b)\\ &=\left\{v_1\frac{\partial}{\partial a}+\frac{1}{8}v_1{}^3\frac{\partial^3}{\partial a^3}+\frac{1}{192}v_1{}^5\frac{\partial^5}{\partial a^5}+\ldots.\right\}\\ &\times\left\{v_2\frac{\partial}{\partial b}+\frac{1}{8}v_2{}^3\frac{\partial^3}{\partial b^3}+\frac{1}{192}v_2{}^5\frac{\partial^5}{\partial b^5}+\ldots.\right\}\mathrm{F}(a,\ b)\\ &=v_1v_2\frac{\partial^2}{\partial a\partial b}\mathrm{F}(a,b)\\ &\quad+\frac{1}{8}v_1v_2\left\{v_1{}^2\frac{\partial^4}{\partial a^3\partial b}\mathrm{F}(a,b)+v_2{}^2\frac{\partial^4}{\partial a\partial b^3}\mathrm{F}(a,b)\right\}+\ldots. \end{aligned}$$

Summary.

A modified form of Taylor's Theorem is derived which allows functions of cosines to be developed directly in series of cosines of multiple angles. This is applied to the determination of the modulation frequencies in a non-linear device such as the thermionic valve.

The author desires to tender his acknowledgment to the General Electric Company and the Marconiphone Company, on whose behalf the work was done which led to this publication.

LXXIII. *Notices respecting New Books.*

Actualités Scientifiques et Industrielles.
Hermann et Cie, Paris.

No. 35. *Premiers Essais de Cinematographie ultra-rapide.* N. A. Magran. [Pp. 28, figs. 32.] (Prix 15 fr.)

A form of cinematograph camera is described taking from two to three thousand views per second. Methods of ultra-rapid cinematography are discussed.

No. 36. *Probabilités et Morphologie.* A. Sainte-Laguë. [Pp. 30, figs. 7.] (Prix 6 fr.)

This lecture treats of the mathematical laws of probability and their application to certain biological problems of growth, of morphology, and of flight.

No. 37. *Influence des facteurs électriques sur la végétation.* N. Marinesco. [Pp. 27, figs. 9.] (Prix 7 fr.)

This gives a theoretical and experimental enquiry into the effect of external electrical fields upon the movements of fluids in plants and the ascent of sap.

No. 46. *Cinematographie jusqu'a* 12,000 *vues par seconde.* A. Magnan. [Pp. 19, figs. 20.] (Prix 15 fr.)

An improved form of ultra-rapid cinematograph camera has been employed to study the flight of insects. A table of data is given.

No. 60. *Le vol au point fixe.* A. Magnan and A. Sainte-Laguë. [Pp. 31, figs. 5.] (Prix 10 fr.)

The hovering flight of flies has been studied and analyzed by methods of ultra-rapid cinematography. Certain Diptera make 120 wing-beats per second.

No. 64. *Les phénomènes biologiques dans le cadre des sciences exactes.* T. Cahn. [Pp. 20.] (Prix 6 fr.)

The author extends the mechanistic basis of biology to growth and heredity by assuming that every experience produces an indelible trace in the organism. Physico-chemical and biological examples are cited.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Fig. 1.

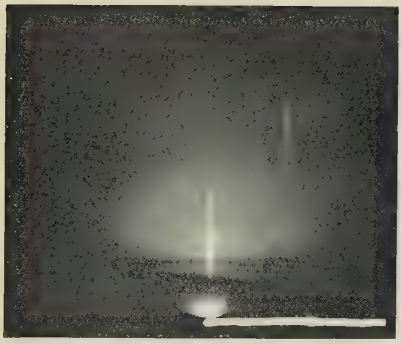


Fig. 2.

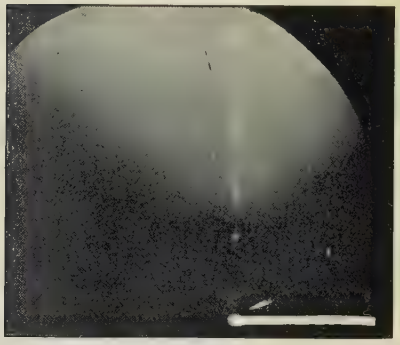


Fig. 3.

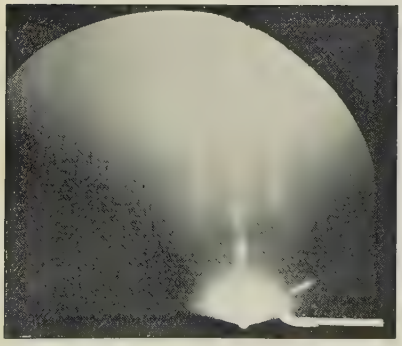


Fig. 4.

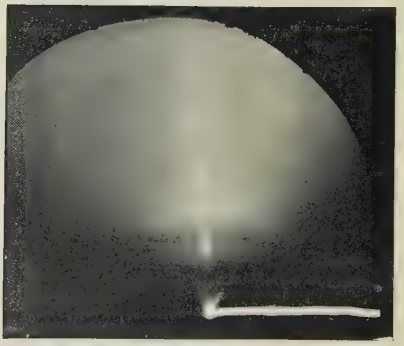


Fig. 5.

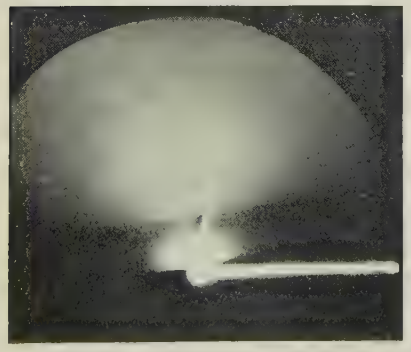


Fig. 6.

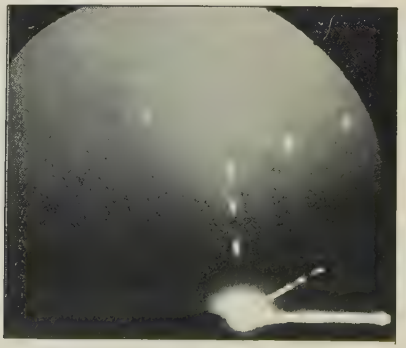
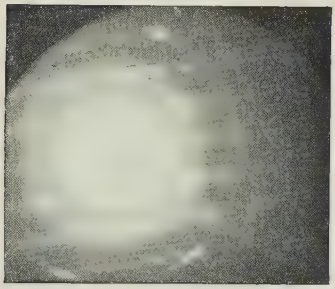


Fig. 1.



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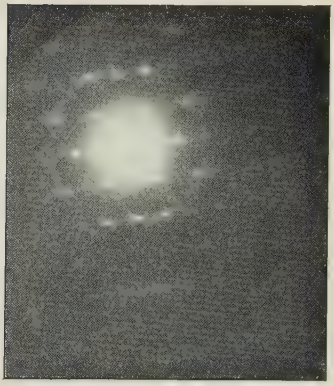


Fig. 3.

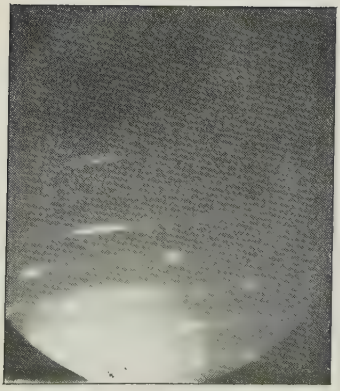


Fig. 4.

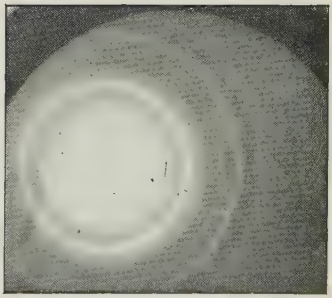
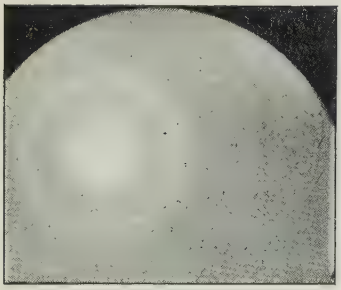


FIG. 5.



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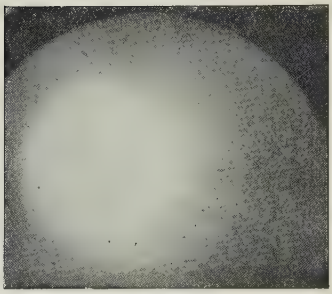


FIG. 7.

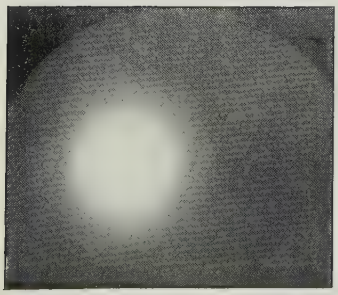
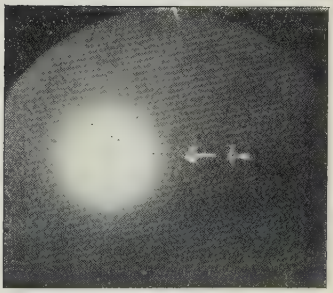


FIG. 8.



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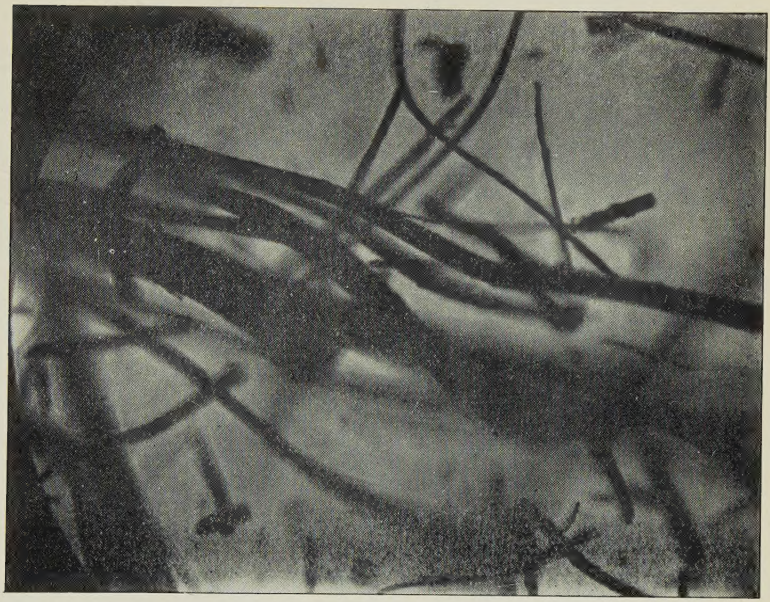


FIG. 10.

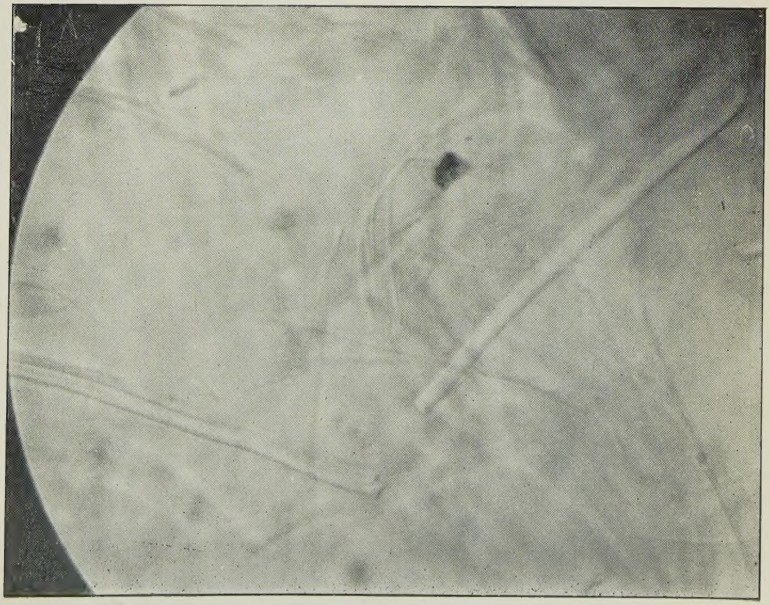
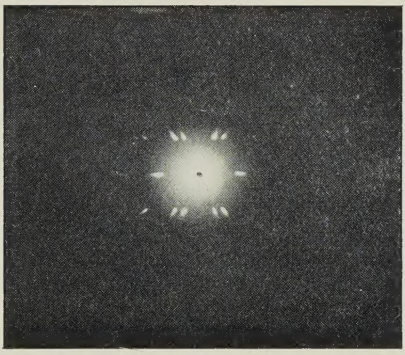
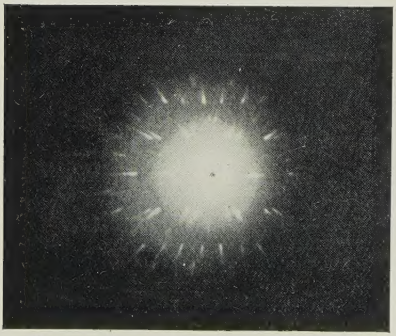


FIG. 1.



Celluloid Diffraction Pattern. Central portion.

FIG. 2.



Celluloid Diffraction Pattern. Outer portion.